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ABSTRACTS

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List of 45-minute Invited Lecturers:
O. N. Ageev, C. Bandle, C. Bardos, J. Batt, H. G. Bock, V. M. Buchstaber, I. Capuzzo Dolcetta, A. A. Davydov, M. del Pino, S. Yu. Dobrokhotov, Yu. A. Dubinskii, A. Favini, A. V. Fursikov, R. V. Gamkrelidze, H. Ishii, W. Jäger, V. Yu. Kaloshin, E. Ya. Khruslov, A. Kozhevnikov, V. V. Kozlov, S. B. Kuksin, S. P. Novikov, P. I. Plotnikov, A. G. Sergeev, M. B. Sevryuk, I. Shafrir, A. E. Shishkov, V. D. Stepanov, T. A. Suslina, I. A. Taimanov, A. Tesei, D. V. Treschev, L. Veron, H.-O. Walther, J. Wei.

Extensions of Typical Group Actions

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Typical dynamical systems is an interesting and popular subject to study in modern dynamics. Breaking the framework of "smoothness" reveals many natural dynamic properties that are sometimes surprisingly common. We will be mostly focused on the topic how to extend a typical action to an action of "the larger" group with a prescribed sequence of dynamic properties. In particular, for every countable Abelian group G, we find the set of all its subgroups H ($H \leqslant G$) such that a typical measure-preserving H-action on a standard atomless probability space $(X, \mathcal{F}, \mu)$ can be extended to a free measure-preserving G-action on $(X, \mathcal{F}, \mu)$. The description of all such pairs $H \leqslant G$ is given in purely group terms, in the language of the dual $\widehat{G}$ and G-actions with discrete spectrum. As an application, we answer the question of when a typical H-action can be extended to a G-action or to a G-action with a certain dynamic property. In particular, examples of pairs $H \leqslant G$ such that G is countable Abelian and a typical H-action is not embeddable in a G-action are presented for the first time.

Hypoellipticity for a New Class of Infinitely Degenerate Elliptic Operators[1]

T. M. Akhunov
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Elliptic differential equations are a natural generalization of the Laplace equation and are among the most studied differential equations. From an analytical perspective, one of the key properties for these equations is the regularity of their solutions. A classical result from the PDE theory is that solutions to second order uniformly elliptic operators (which at every fixed point resemble the Laplacian in a uniform way) are 2 derivatives smoother than the data in most Hölder and Sobolev spaces. The weakest form of this property, called hypoellipticity, is when smooth data lead to smooth solutions. For second order operators, the amount of gain may be less than 2. Moreover, the local form of this property fails in very degenerate cases. In his landmark 1964 paper, Lars Hörmander established a bracket criterion for degenerate elliptic operators that gain a positive number of derivatives. In this talk, I will discuss a new class of examples of degenerate elliptic operators that gain no derivatives. This work generalizes previous work of Fedii, Morimoto and J. J. Kohn.

[1]Joint work with Cristian Rios (University of Calgary).

Changing the Predictor of the Extended and Modified Extended Backward Differentiation Formula

E. Alberdi Celaya, J. J. Anza Aguirrezabala
The University of the Basque Country, Leioa, Spain

One of the directions carried out in the search of higher order and more stable methods when solving Initial Value Problems (IVPs) has been the use of off-step points called superfuture points. Cash has introduced methods using superfuture points to solve stiff IVPs. These methods are known as Extended Backward Differentiation Formulae (EBDF) and Modified Extended BDF (MEBDF). They use two BDF predictors and one implicit multistep corrector. Both methods are A-stable up to order 4 and $A(\alpha)$-stable up to order 9, and the class MEBDF has better stability properties than the class EBDF.

We have modified the EBDFs and MEBDFs using the Numerical Differentiation Formulae (NDFs) as predictors. In this way two families of methods (the EBDF and the MEBDF families) have been obtained. They are A-stable up to order 4, and in order 5 some of the new methods have better stability properties than the methods of origin. We will present the construction of both families and their stability and accuracy properties. A comparison between the two families is done, showing that the methods of the MEBDF family have better stability characteristics and they result more accurate than their respectives in the EBDF family.

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On Interior Regularity of Solutions to Zakharov–Kuznetsov Equation

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We consider the initial-boundary value problem in a domain $\Pi_T^+ = (0,T)\times\mathbb{R}^2_+$, where $T>0$ is arbitrary and $\mathbb{R}^2_+ = \{(x,y) : x>0\}$, for the Zakharov–Kuznetsov equation

$$u_t + u_{xxx} + u_{xyy} + uu_x = 0$$

with the initial and the boundary conditions

$$u(0,x,y) = u_0(x,y), \quad u(t,0,y) = u_1(t,y).$$

This equation is one of two-dimensional generalizations of Korteweg–de Vries equation and describes nonlinear waves in dispersive media, propagating in x-direction with deformations in the transversal y-direction.

Let u_0 belong to the space $L^{\alpha}_{2,+} = \{\varphi : (1+x)^{\alpha}\varphi \in L_2(\mathbb{R}^2_+)\}$ for some positive α while $u_1 \in H^{s/3,s}((0,T)\times\mathbb{R}))$ for $s > 3/2$. The existence of a weak solution to this problem in the space

$$u \in L_{\infty}(0,T; L^{\alpha}_{2,+}), \qquad u_x, u_y \in L_2(0,T; L^{\alpha-1/2}_{2,+})$$

was proved by A. V. Faminskii. Moreover, if $\alpha \geqslant 1$, then a solution is unique.

Here we establish the property of interior regularity of these solutions.

Theorem 1. *Let $u_0 \in L^{\alpha}_{2}$ for $\alpha \geqslant 1/2$ and $u_1 \in H^{2/3,2}((0,T)\times\mathbb{R}))$. Then we have $u_x, u_y \in L_{\infty}(\delta, T; L^{\alpha-1/2}_{2,+})$ for any $\delta \in (0,T)$.*

Moreover, the gain of regularity of these solutions strictly inside Π^{+}_{T} with respect to the rate of decrease of the initial function u_0 as $x \to +\infty$ is obtained. In particular, these solutions can possess derivatives of any prescribed order.

On Regularity Properties of Solutions to Hysteresis-Type Problems

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We consider the initial-boundary value problem for a parabolic equation with a discontinuous hysteresis operator on the right-hand side in the equation. Problems of this kind describe the so-called processes “with memory” in which various substances interact according to the hysteresis law.

In the case of one-dimensional space variable, we discuss the optimal regularity results for solutions, i.e., the L^{∞}-estimates for the second-order space derivative u_{xx} and for the first-order time derivative u_t.

The talk is based on the results obtained in collaboration with Nina Uraltseva. This work was supported by the Russian Foundation for Basic Research (RFBR), grant number 14-01-00534, and by the St. Petersburg State University grant 6.38.670.2013.

W-Algebras and Higher Analogs of the Kniznik–Zamolodchikov Equations

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The key role in the derivation of the Knizhnik–Zamolodchikov equations in the WZW-theory is played by the energy-momentum tensor. The last one is constructed from a central second-order Casimir element in the universal enveloping algebra of a corresponding Lie algebra. In the paper, we investigate the possibility of constructing analogs of Knizhnik–Zamolodchikov equations using higher-order central elements. Third-order Gelfand elements for a simple Lie algebra of series A and fourth-order Capelli elements for a simple Lie algebra of series B and D are considered. In the first case, the construction is not possible while the desired equation is obtained in the second case.

Estimate of the Decay Exponent of an Operator Semigroup Associated with a Second-Order Linear Differential Equation

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In a Hilbert space H, we consider a second-order linear differential equation

$$u''(t) + Du'(t) + Au(t) = 0$$

and the corresponding quadric pensil $L(\lambda) = \lambda^2 I + \lambda D + A$ with a positive definite operator A. By H_s we denote the collection of Hilbert spaces generated by the operator $A^{1/2}$, $\| \cdot \|_s$ is the norm on H_s. We will assume that D is a bounded operator acting from H_1 to H_{-1} and

$$\inf_{x \in H_1, x \neq 0} \frac{\mathrm{Re}(Dx, x)_{-1,1}}{\|x\|_1^2} = \delta > 0.$$

(here $(\cdot, \cdot)_{-1,1}$ is the duality pairing on $H_{-1} \times H_1$). The second-order differential equation can be considered as a system $w'(t) = \mathcal{T} w(t)$ in the "energy" space $H \times H_1$, where

$$w(t) = \begin{pmatrix} u'(t) \\ u(t) \end{pmatrix}, \quad \mathcal{T} = \begin{pmatrix} -D & -A \\ I & 0 \end{pmatrix}.$$

We estimate an exponential decay rate for the semigroup generated by the operator $\mathcal{T}$ in the space $H \times H_1$. We also obtain localization of the spectrum of the pencil $L(\lambda)$, showing that

$$\sigma(L) = \sigma(\mathcal{T}) \subset \{\lambda \in \mathbb{C} \mid \mathrm{Re}\lambda \leqslant -\omega, |\mathrm{Im}\lambda| \leqslant \kappa(b)|\mathrm{Re}\lambda| + b\}$$

for some positive ω and for all $b > 0$.

Integral Equations of Convolution Type with Monotone Nonlinearity

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For the nonlinear integral convolution type equations

$$\lambda \cdot F\left[x, u(x)\right] + \int_0^1 \varphi\left(|x-t|\right)\, u(t)\, dt = f(x)\,,$$

$$u(x) + \lambda \cdot \int_0^1 \varphi\left(|x-t|\right)\, F\left[t, u(t)\right]\, dt = f(x)\,,$$

$$u(x) + \lambda \cdot F\left[x, \int_0^1 \varphi\left(|x-t|\right)\, u(t)\, dt\right] = f(x)\,,$$

the existence and the uniqueness theorems and estimates for solutions in the real spaces $L_p(0,1)$, $1 < p < \infty$, were proved in [1] without any restrictions on the absolute value of the parameter λ.

In the present work, we use the theory of monotone operators to prove that in the case of the space $L_2(0,1)$, these solutions can be found by the method of Picard type successive approximations, where the absolute value of the parameter λ need not to be "small". In contrast to [2], where similar equations with kernels of potential type on the real axis were considered, here we employ the method of potential monotone operators to construct new successive approximations and improve substantially the estimates for the rate of convergence. Using the gradient method (the method of steepest descent), we succeed to solve equations with power nonlinearities both in $L_p(0,1)$ and the weighted spaces $L_p(\rho)$. This issue is not covered by the results obtained in [2].

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Abstract Euler–Poisson–Darboux Equation with Nonlocal Condition

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The correct formulation of the initial conditions for the Euler–Poisson–Darboux equation

$$u''(t) + \frac{k}{t}u'(t) = Au(t), \quad k > 0,$$

is as follows (see [1]):

$$u(0) = u_0, \quad u'(0) = 0.$$

We found that conditions to be imposed on the operator A let the initial condition $u(0) = u_0$ be replaced by a nonlocal condition of the form

$$\lim_{t \to 1} I_{\nu,\beta} u(t) = u_1, \qquad \nu = \frac{k-1}{2}, \quad \beta > 0,$$

or of the form

$$\lim_{t \to 1} I_{0+}^{\beta} u(t) = u_2, \qquad \beta > 0,$$

where $I_{\nu,\beta}$ is the Erdelyi–Kober operator (see [2, § 18]) and I_{0+}^{β} is the left-sided Riemann–Liouville integral (see [2, § 2]).

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Semi-Linear Elliptic Equations in Spaces of Constant Curvature

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We consider problems of the type $\Delta u + \lambda u + u^p = 0$ in a ball in $\mathbb{S}^n$ or in $\mathbb{H}^n$. Here Δ is the Laplace–Beltrami operator, $\lambda \in \mathbb{R}^+$ and $p > 1$. We study the existence of positive solutions satisfying Dirichlet boundary conditions. A fairly complete picture of the radial solutions will be presented and some non radial solutions bifurcating from the eigenvalues of the linear problem will be discussed. The talk reports on a project in collaboration with Y. Kabeya (Osaka Prefecture University) and H. Ninomiya (Meiji University).

The Vlasov–Dirac–Benney Equation at the Cross Road of Spectral Analysis and Integrability

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N. Besse
Jean Lamour Institute, Nancy, France

This talk concerns an equation which is dubbed Vlasov Dirac Benney because on the one hand, it is a Vlasov equation with the usual potential replaced by the Dirac distribution, and on the other hand, it is equivalent in some cases to an equation derived by Benney for long wave approximation.

With the presence of this Dirac mass, the problem exhibits some drastic instabilities.

However, almost necessary and sufficient conditions for local in time stability may be given.

As observed by several authors, e.g., Zakharov [1], the Benney equation belongs to the class of integrable systems.

Hence it has many formal properties but what is striking is that the case where these properties turns out to be "rigorous" in the sense of functional analysis corresponds to the above stability theorems.

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Harmonic and Spectral Analysis of Operators with Bounded Degrees and Bounded Semigroups of Operators

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Let $\mathcal{X}$ be a complex Banach space and $\operatorname{End}\mathcal{X}$ be the Banach algebra of all bounded linear operators acting on $\mathcal{X}$. Let an operator $T \in \operatorname{End}\mathcal{X}$ have bounded powers (i.e. $\sup\limits_{n\geqslant 1}\|T^n\| < \infty$) and

$$\sigma(T) \cap \{\lambda \in \mathbb{C} : |\lambda| = 1\} = \{\gamma_1, \gamma_2, \dots \gamma_m\}.$$

We obtain the following results.

Theorem 1. *The operator T has the asymptotic representation*

$$T^n = \sum_{k=1}^{m} \gamma_k^n A_k(n),$$

where operators $A_k : \mathbb{Z}_+ \to \operatorname{End}\mathcal{X}$, $1 \leqslant k \leqslant m$, have the following properties:

1. *the operators* $A_k(n)$*,* $n \geqslant 1$*,* $i \leqslant k \leqslant m$*, belong to the minimal closed subalgebra of* $\operatorname{End}\mathcal{X}$ *generated by* T;

2. $\lim\limits_{n\to\infty} \|A_k(n+1) - A_k(n)\| = 0$;

3. $\lim\limits_{n\to\infty} \|TA_k(n) - \gamma_k A_k(n)\| = 0$;

4. $\lim\limits_{n\to\infty} \|A_k(n)A_p(n)\| = 0$ *for* $k \neq p$*,* $1 \leqslant k,\ p \leqslant m$*.*

An analogous representation is obtained for C_0-semigroups.

Theorem 2. *Let* $T : \mathbb{R}_+ \to \operatorname{End}\mathcal{X}$ *be a bounded strongly continuous semigroup of operators,* A *be its generator, and the spectrum of* A *be such that*

$$\sigma(A) \cap (i\mathbb{R}) = \{i\lambda_1, \ldots, i\lambda_m\}.$$

Then the semigroup T *can be represented as*

$$T(t) = \sum_{k=1}^{m} B_k(t)\mathrm{e}^{i\lambda_k t} + B_0(t), \quad t \geqslant 0,$$

where functions $B_k : \mathbb{R} \to \operatorname{End}\mathcal{X}$ *are continuous with respect to the uniform operator topology. These functions have the following properties*:

1. *the operators* $B_k(t)$*,* $t_0 \leqslant t$*,* $0 \leqslant k \leqslant m$*, belong to the minimal closed subalgebra of* $\operatorname{End}\mathcal{X}$ *generated by the operators* $T(t)$*,* $t \geqslant t_0$*;*

2. $\lim\limits_{t\to\infty} \|B_0(t)\| = 0$*;*

3. *the functions* B_k*,* $1 \leqslant k \leqslant m$*, are slowly varying at infinity and can be extended to* $\mathbb{C}$ *as entire functions of exponential type with* $\lim\limits_{t\to\infty} \|B_k'(t)\| = 0$*;*

4. $\lim\limits_{t\to\infty} \|T(t)B_k(t) - \mathrm{e}^{i\lambda_k t}B_k(t)\| = 0$.

5. $\lim\limits_{t\to\infty} \|B_k(t)B_p(t)\| = 0,\ \ k \neq p$.

Stationary Solutions of the Vlasov–Poisson System: Methods of Construction and New Results

J. Batt
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The Vlasov–Poisson system describes the evolution of mass systems due to the attracting or repulsing forces between the particles (stellar dynamical or plasmaphysical case, respectively).

It reads

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{\partial U}{\partial x}\frac{\partial f}{\partial v} = 0,$$
$$\Delta U(t,x) = \pm 4\pi \varrho(t,x), \quad \varrho(t,x) := \int f(t,x,v)dv,$$

where $f = f(t,x,v)$ is the distribution function, $U = U(t,x)$ the gravitational or electric potential and $\varrho = \varrho(t,x)$ the local density in $\mathbb{R}^3$.

The lecture presents various ways of constructing stationary (i.e., t-independent) solutions: the (classical) direct method, the inverse problem (which leads to an ill-posed problem of the Radon transform), and the variation method.

A new way allows the construction of new singular, so-called "flat" solutions that are δ-distributed in x_3 and v_3. This is a joint work with E. Jörn and Y. Li.

Properties of Meromorphic Solutions of Homogeneous and Non-Homogeneous Higher-Order Linear Differential Equations

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In this paper, we study the growth of meromorphic solutions of homogeneous and non-homogeneous higher order linear differential equations

$$f^{(k)} + \sum_{j=1}^{k-1} B_j f^{(j)} + B_0 f = 0 \quad (k \geqslant 2),$$

$$f^{(k)} + \sum_{j=1}^{k-1} B_j f^{(j)} + B_0 f = F \quad (k \geqslant 2),$$

where $B_j(z)$, $j = 0, 1, \cdots, k-1$, and $F(z)$ are meromorphic functions of finite iterated $p-$order. We show under certain conditions on the coefficients that all meromorphic solutions $f \not\equiv 0$ of the above equations have an infinite iterated $p-$order and an infinite iterated lower $p-$order. Furthermore, we give some estimates for the iterated exponent of convergence. We improve the results due to Chen; Shen and Xu; He, Zheng and Hu, and others.

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Metastable Patterns in Solutions of Chafee–Infante's Problem

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We study the slow evolution of patterns in solutions of the singulary perturbed Chafee–Infante's problem [1]

$$\begin{aligned} &u_t = \mu^2 u_{xx} + u - u^3, \quad 0 < x < \pi, \quad t > 0, \\ &u(0,t) = 0, \quad u(\pi,t) = 0. \end{aligned} \tag{1}$$

Equation (1) defines a nonlinear semiflow $S^t : H_0^1(0,\pi) \to H_0^1(0,\pi)$, $t \geqslant 0$. This semiflow admits a global attractor A_μ. The set of equilibria of (1) E_μ, which is the solution set to the boundary value problem

$$\mu^2 u_{xx} + u - u^3 = 0, \quad u(0) = u(\pi) = 0, \tag{2}$$

depends of $\mu > 0$. If $\mu < 1$, then the homogeneous equilibria is unstable. If $\frac{1}{n+1}\mu < \frac{1}{n}$, then E_μ contains exactly n pairs of nonconstant equilibria u_k, $\hat{u_k} = -u_k$, $k = 1, ..., n$. The equilibria u_k, $\hat{u_k}$ bifurcate from zero equilibria and have exactly $k-1$ zeros of $x = \frac{\pi}{2(k-1)}, \frac{3\pi}{2(k-1)}, \ldots, \pi - \frac{\pi}{2(k-1)}$ on $(0,\pi)$. The equilibria u_1, $\hat{u_1}$ are stable, while u_k, $\hat{u_k}$, $k = 2, \ldots, n$ are unstable, and the dimension of unstable manifolds of u_k, $\hat{u_k}$ is exactly $k-1$. As $\mu \to 0$, u_k approaches on $(0,\pi)$ a step function with values $1, -1, 1, \ldots$ and jumps at the zeros of u_k.

Thus, for large t, typical solutions of (1) approach u_1 or $\hat{u_1}$. For μ small, however, a pattern of interfacial layers typically forms far from stable equilibriums similarly to [2, 3]. These patterns persist for extremely long times. The solution $u(x,t)$ of (1) changes extremely slowly if $u(x,0)$ is an approximate solution to problem (2) for the interior transitional layer. We construct approximate solutions to (2) using Galerkin's method.

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On Dynamics of the Trace Map[1]

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We consider the trace map

$$F(x,\, y) = (xy,\, (x-2)^2) \tag{1}$$

[1] This is a joint work with L. S. Efremova.

on the plane, where $(x,\, y)$ is a point of the plane $\mathbb{R}^2$.

It is shown in [1] that the study of passing and reflection coefficients of the plane wave with a given impulse in the field of the crystal lattice with knots, which form Thue–Morse chain, can be reduced to the study of the trace map conjugate with map (1).

We distinguish the following sets on the plane xOy:

(1) the closed triangle $\Delta = \{(x; y) \in \mathbb{R}^2 : x, y \geqslant 0,\ x + y \leqslant 4\}$;

(2) the unbounded set $G_\Delta = \{(x; y) \in \mathbb{R}^2 : x, y \geqslant 0,\ x + y \geqslant 4\}$;

(3) the unbounded set $\widetilde{G} = G_\Delta \cap (\bigcup\limits_{i=0}^{+\infty} F^{-i}(D_{+\infty})$, where $F^{-i}(D_{+\infty})$ is the i-th complete preimage of the set $D_{+\infty} = \{(x; y) \in \mathbb{R}^2 : x \geqslant 3, y \geqslant 1\}$;

(4) the unbounded set $G' = G_\Delta \setminus int\, \widetilde{G}$, where $int\,(\cdot)$ denotes the interior of a set.

The following theorem contains the main result concerning map (1).

Theorem 1 (see [2])**.** *The nonwandering set $\Omega(F)$ of the map F is the union of the triangle Δ and the perfect set G' nowhere dense in G_Δ and possessing the following properties*:

1. *the set G' is the union of unbounded curves that define F-completely invariant local lamination of codimension* 1 *in the set $int\, G_\Delta$; moreover, the set of algebraic curves is everywhere dense in the set G'*;

2. *the map $F|_\Delta$ is topologically mixing, and its periodic points are everywhere dense in Δ.*

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Probabilistic Interpretation of Solutions to the Cauchy Problem for Systems of Nonlinear Parabolic Equations

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Probabilistic approach in the study of systems of nonlinear parabolic equations allows one to analyze the structure of these systems and obtain more information about various types of solutions (classical, generalized or viscosity) to the Cauchy problem. We consider two types of quasilinear PDE systems. One of them is marked by the diagonal entry of second derivatives,

$$\frac{\partial u_l}{\partial s} + [Lu]_l + \sum_{m=1}^{d_1}\sum_{k=1}^{d} B^k_{lm}\nabla_k u_m + \sum_{m=1}^{d_1} c_{lm}u_m + g_l = 0, \quad u_l(T,x) = u_{0l}(x), \qquad (1)$$

where $[Lu]_l = \frac{1}{2}TrA^*\nabla^2 u_l A + \langle a, \nabla u_l\rangle$ and all coefficients depend on x, u, and ∇u. We can modify (1) slightly to have $A = A_l(x, u, \nabla u_l)$ and $a = a_l(x, u, \nabla u_l)$ but $B^k_{lm} \equiv 0$.

We address on a classical solution to (1) in a semilinear case first. To construct its probabilistic representation based on a standard Wiener process $w(t) \in R^d$ on a probability space $(\Omega, \mathcal{F}, P)$, we consider the following system of stochastic equations:

$$d\xi(t) = a(\xi(t), u(t, \xi(t)))dt + A(\xi(t), u(t, \xi(t)))dw(t), \quad \xi(s) = x, \qquad (2)$$

$$d\eta(t) = c(\xi(t), u(t, \xi(t)))\eta(t)dt + C(\xi(t), u(t, \xi(t)))(\eta(t), dw(t)), \quad \eta(s) = h, \qquad (3)$$

$$\langle h, u(s,x)\rangle = \mathbf{E}_{s,x,h}\left[\langle \eta(T), u_0(\xi(T))\rangle + \int_s^T \langle \eta(\theta), g(\xi(\theta), u(\theta, \xi(\theta)))\rangle d\theta\right]. \qquad (4)$$

Given $B = CA$, system (2)–(4) composes a probabilistic counterpart of problem (1), at least in the case where classical solutions are considered. Namely, assuming that there exists a classical solution to (1), we can prove that (4) gives its probabilistic representation. On the other hand, if the coefficients and the initial data provide the existence of a unique solution to system (2)–(4) and, moreover, the function u given by (4) is proved to be C^2-smooth, then (4) defines a unique classical solution of (1). Under suitable conditions, these results can be generalized to quasilinear and even fully nonlinear parabolic systems by using the method of differential prolongation.

To construct a probabilistic representation for a generalized solution to (1), one has to deal with similar stochastic equations with respect to a time reversal process $\hat{\xi}(t)$. Finally, as soon as viscosity solutions are considered, one leaves equations (2) and (3) unchanged replacing (4) by the so-called backward stochastic equation

$$dy(t) = -g(\xi(t), y(t), z(t))dt + z(t)dw(t), \quad y(T) = u_0(\xi(T)),$$

for $y(t) = u(t, \xi(t)), z(t) = A^*\nabla u(t, \xi(t))$.

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The second type of quasilinear parabolic systems is the so-called fully coupled systems. A typical example is a system describing the cell growth under contact inhibition and has the form

$$u_t = \frac{1}{2}\Delta[u + uv] + (1 - \beta_1 u - \alpha v)u, \quad u(0, x) = u_0(x) \tag{5}$$

$$v_t = \frac{1}{2}\Delta[v + uv] + \gamma(1 - \beta_2 u - \kappa v)v, \quad v(0, x) = v_0(x) \tag{6}$$

where $\alpha, \beta_1, \beta_2, \gamma, \kappa$ are constants.

We construct a probabilistic representation for a generalized solution to (5)–(6) as well. To this end, we use a dual PDE system to define a suitable stochastic process along with its time reversal, and crucially exploit some results of the stochastic flow theory for an SDE of form (2).

A Priori Estimates and Initial Trace for Hamilton–Jacobi Equation with Gradient Absorption Terms

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Here we consider the nonnegative solutions of the parabolic Hamilton–Jacobi equation

$$u_t - \Delta u + |\nabla u|^q = 0$$

in $\Omega \times (0, T)$, where $\Omega = \mathbb{R}^N$ or Ω is a bounded domain of $\mathbb{R}^N$, $q > 0$.

We give new local and global a priori estimates for solutions without conditions as $|x| \to \infty$ or $x \to \partial\Omega$, and the corresponding existence results with an initial data measure $u_0 \in \mathcal{M}^+(\Omega)$.

We also study the existence of an initial trace. We show that all the solutions admit a trace as a Borel measure (S, u_0). This means that there exist a set $\mathcal{S} \subset \Omega$ such that $\mathcal{R} = \Omega \backslash \mathcal{S}$ is open, and a (possibly unbounded) measure $u_0 \in \mathcal{M}^+(\mathcal{R})$, such that

$$\lim_{t\to 0} \int_{\mathcal{R}} u(., t)\psi = \int_{\mathcal{R}} \psi du_0, \qquad \forall\psi \in C_c^0(\mathcal{R}),$$

$$\lim_{t\to 0} \int_{\mathcal{U}\cap\mathcal{S}} u(., t)dx = \infty, \qquad \forall\mathcal{U} \text{ open} \subset \Omega, \text{s.th. } \mathcal{U} \cap \mathcal{S} \neq \varnothing.$$

We give more general existence results for solutions with such a trace, according to assumptions on u_0, and establish their behavior as $|x| \to \infty$. In particular, we construct a solution with the trace $(\mathbb{R}^{N+}, 0)$. When $q \leqq 1$, we show that $\mathcal{S}$ is empty.

Parabolic Equation with Small Parameter in Boundary Condition

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Let Ω be a domain in $\mathbb{R}^n$, $n \geqslant 2$, and $\Sigma = \partial\Omega \times (0,T)$.

The boundary value problem for a parabolic equation in $\Omega \times (0,T)$ is considered. The boundary condition on Σ contains a small coefficient $\varepsilon > 0$ of the principal term, which is the time derivative of u.

The existence of a unique solution is proved in a Hölder space, and the coercivity estimate uniform with respect to ε is obtained. As $\varepsilon \to 0$, the convergence of solutions to the solution of the unperturbed problem is established.

Ergodic Properties of a Chaotic Collective Walk Governed by Anosov Maps

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We consider an infinite collection of particles moving independently on a unit d-dimensional torus according to some local chaotic dynamics and interacting only if they are close enough to each other (e.g. balls in Sinai billiard). We reduce this infinite dimensional system to the so-called self-consistent one acting on the d-torus but depending on both current spatial position and current measure. In other words, the self-consistent system acts in the space of measures on the d-torus nonlinearly in the same manner as a stochastic nonlinear Markov chain does. For the local dynamics defined by Anosov diffeomorphisms, we give sufficient conditions under which in the limit of weak interactions the collective walk either breaks into independent components or demonstrates synchronization phenomenon depending on fine features of interactions.

Boltzmann Equation and Hydrodynamics at the Burnett Level

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We present a review of some results on Burnett-type hydrodynamic equations derived from the Boltzmann equation. The well-known problem here is connected with regularization of classical (ill-posed) Burnett equations [1–5] . There are several ways to deal with this problem. We discuss in detail one of the approaches, proposed in [1] and further developed in [2–4]. Our approach is based on infinitesimal changes of variables, it shows that the way of truncation of the Chapman–Enskog series is not unique. It is the only approach that does not use any information beyond the classical Burnett equations. We show how to derive a two-parameter family of stable generalized Burnett equations (GBEs) [2] and discuss the optimal choice of the parameters. Surprisingly, the resulting well-posed equations are simpler than the original Burnett equations. The equations are derived for arbitrary intermolecular forces. Some special properties of (a) stationary problems and (b) linear non-stationary problems are discussed in more detail. Finally, we present some recent results on the shock-wave structure [4], which show that GBEs yield certain improvement of the Navier-Stokes results for moderate Mach numbers. Some open questions are also discussed.

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The Spectral Problem in the Waveguide with Bi-isotropic Filling[1]

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The development of effective algorithms for calculating electromagnetic field in metamaterials, including artificial bi-isotropic materials, is an important problem because these algorithms can be used for solving inverse problems of creating the material with necessary properties. One of the widely used numerical methods is the method of finite elements, but this method can give nonphysical solutions (spurious modes) for the electromagnetic problems in full vector statement. In this paper, we present the generalized statement of a spectral problem in a waveguide with bi-isotropic filling that prevents the appearance of spurious modes.

Consider a waveguide with axis Oz and cross section S having perfectly conducting walls and partially constant bi-isotropic filling. The constitutive relations in the frequency domain are

$$\mathbf{D} = a^p_{11}\mathbf{E} + a^p_{12}\mathbf{H}, \quad \mathbf{B} = a^p_{21}\mathbf{E} + a^p_{22}\mathbf{H}, \quad (x, y) \in S_p, \quad p = 1, ..., P,$$

where a^p_{11}, a^p_{12}, a^p_{21}, a^p_{22} are constants, $a^p_{11}a^p_{22} - a^p_{12}a^p_{21} \neq 0$, $S = \bigcup_{p=1}^{P} S_p$. Taking into account the Maxwell's equations, the constitutive relations, and the conjugation conditions, we propose the following generalized statement of the spectral problem for

[1]The work is supported by the Russian Foundation for Basic Research (grant 06-01-00146-a).

the field harmonically depending on time ($e^{-i\omega t}$): one must find the propagation constants γ and corresponding vector eigenfunctions $\mathbf{E}(x,y)$ whose components belong to the Sobolev space $W_2^1(S)$, satisfy the equation

$$\int\limits_S \frac{1}{a_{22}}\left((\mathrm{rot}_\perp \mathbf{E}, \mathrm{rot}_\perp \mathbf{E}^*) + 2\mathrm{div}_\perp \mathbf{E}\cdot \mathrm{div}_\perp \mathbf{E}^*\right) ds + \gamma^2 \int\limits_S \frac{1}{a_{22}}\left((\mathbf{E}, \mathbf{E}^*) + E_3 E_3^*\right) ds$$
$$+ i\gamma \int\limits_S \frac{1}{a_{22}}\left((\nabla_\perp E_3, \mathbf{E}^*) - (\mathbf{E}, \nabla_\perp E_3^*) - 2\mathrm{div}_\perp \mathbf{E}\cdot E_3^* + 2E_3\cdot \mathrm{div}_\perp \mathbf{E}^*\right) ds$$
$$- ik \int\limits_S \left(\frac{a_{21}}{a_{22}}(\mathbf{E}, \mathrm{rot}_\perp \mathbf{E}^*) - \frac{a_{12}}{a_{22}}(\mathrm{rot}_\perp \mathbf{E}, \mathbf{E}^*)\right) ds$$
$$+ \gamma k \int\limits_S \frac{a_{21}-a_{12}}{a_{22}}\left([\mathbf{e}_z, \mathbf{E}], \mathbf{E}^*\right) ds - k^2 \int\limits_S \frac{a_{11}a_{22}-a_{12}a_{21}}{a_{22}}(\mathbf{E}, \mathbf{E}^*) ds = 0, \quad k = \frac{\omega}{c}, \tag{1}$$

and such that their restrictions to the boundary ∂S satisfy the condition $[\mathbf{n}, \mathbf{E}]|_{\partial S} = 0$ for any vector-function $\mathbf{E}^*(x,y)$ with the components belonging to $W_2^1(S)$ and satisfying the condition of a perfectly conducting wall. In the equation above, the following notations are used:

$$\nabla_\perp = \left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0\right\}, \;\; \mathrm{rot}_\perp = [\nabla_\perp, \cdot\,], \;\; \mathrm{div}_\perp = (\nabla_\perp, \cdot\,).$$

Consider the Hilbert space $H(S)$ of all vector-functions satisfying the condition $[\mathbf{n}, \mathbf{F}]|_{\partial S} = 0$, with the inner product

$$[\mathbf{F}, \mathbf{G}]|_{H(S)} = \int\limits_S \frac{1}{a_{22}}\left\{(\mathbf{F}, \bar{\mathbf{G}}) + \left(\mathrm{rot}_\perp \mathbf{F}, \mathrm{rot}_\perp \bar{\mathbf{G}}\right) + 2\mathrm{div}_\perp \mathbf{F}\cdot \mathrm{div}_\perp \bar{\mathbf{G}}\right\} ds.$$

In this Hilbert space, equation (1) is equivalent to $L(\gamma)\mathbf{E} = (I + C + \gamma B + \gamma^2 A)\mathbf{E} = 0$, where A, B, and C are compact operators, and the operator A is self-adjoint and positive definite. So the spectrum of problem (1) consists only of the eigenvalues γ_n, among which only a finite number are real.

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Method of Discrete Sources as an Efficient Solver for Scattering of Electromagnetic Waves in Domains with Sharp Corners

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A classical scattering problem in a domain with sharp corners is arisen in optical waveguides in particular. For this case the most of numerical methods encounter difficulties due to singularities at the edges of the corners. To solve such problem effectively, the method of discrete sources (MDS) seems to be the most promising one. The effectiveness of the MDS depends, in its turn, essentially on the way of a source placement. Moreover, for domains with sharp corners the situation becomes dramatically ill-posed.

The model plane scattering problem in the strip region with a sharp ledge is considered. The problem is governed by the Helmholtz equation together with the Neumann condition on the boundary and the radiation condition at infinity. Original ideas about the source allocation for sharp-pointed domains are suggested. An algorithm for obtaining non-smooth solutions, based on the singular value decomposition technique, is presented. The capabilities of the proposed method are demonstrated for the narrow plane channels, where the asymptotic solutions exist.

Portfolio Optimization in the Case of an Asset with a Given Liquidation Time Distribution

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In times of financial crises, many investors have to manage portfolios with low liquidity or illiquid assets, where the paper value of an asset may significantly differ from the actual price proposed by the buyer.

We consider an optimization problem for a portfolio including an illiquid asset, a risky asset, and a riskless asset. In order to find the optimal policies, we work in the optimal consumption framework with continuous time, proposed by Merton. The liquid part of the investment is described by the standard Black–Scholes market, i.e. it contains a riskless bank account and a risky stock which price follows the geometrical Brownian motion. We assume that the illiquid asset is sold at a random moment of

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time with prescribed selling time distribution while generating continuous additional liquid wealth dependent on its paper value. Its paper value is correlated with the the stock price. We also assume that the investor has a HARA-type utility function or, in particular, the logarithmic utility function as a limit case. We study the following two different distributions of the selling time of the illiquid asset: the classical exponential distribution and the more practically relevant Weibull distribution.

The investment problem is to find investment and consumption policies such that the value function of portfolio maximizes the utility of the consumption stream. We obtain the three dimensional Hamilton–Jacobi–Bellman (HJB) equation for the value function. Using the Lie group analysis, we reduce the dimension of the HJB equation. In the case of the classical exponential distribution of the selling time, we reduce HJB equation to an ordinary differential equation. We show under certain conditions the existence and uniqueness of the viscosity solution in both cases. We modify the well known comparison principle for the case of unbounded control and find the bounds for the value function. We also establish smoothness of the viscosity solution for the optimal strategy and provide a closed formulae relevant for numerical calculations. In the exponential case, we find the asymptotic formulae for the optimal policies and show that the optimal solution converges to the Merton closed form solution in the limit of vanishing random income. We provide the results of numerical calculations for the optimal solutions in both cases of selling time distributions, the exponential distribution and the Weibull one.

On the Uniform Resolvent Convergence in the Theory of Boundary Homogenization

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In the two-dimensional strip $\{(x_1, x_2) : 0 < x_2 < \pi\}$, consider the second-order scalar elliptic operator

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} A_{ij} \frac{\partial}{\partial x_j} + \sum_{j=1}^{n} \left(A_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} \overline{A_j} \right) + A_0$$

with sufficiently smooth and bounded coefficients and the singularly perturbed boundary conditions.

We deal with three types of perturbation. The first one is the frequent alternation of boundary conditions. Namely, the lower boundary is partitioned into two families of small segments. The characteristic size of the segments in the first family is of order ε, while the order is $\varepsilon\eta$ for the other family, where $\eta = \eta(\varepsilon)$ is some function. Here ε is a small parameter. It is assumed that the segments in these families alternate. We impose the Robin boundary condition on the first family segments, while the Dirichlét conditions are imposed on the segments of the other family. So, these boundary conditions alternate. The alternation is not necessary to be periodic.

The second type of perturbation is a fast oscillating boundary. Namely, we replace the lower straight boundary of the strip by a fast oscillating one $\{x : x_2 = \eta(\varepsilon)b(x_1/\varepsilon)\}$, where b is a smooth one-periodic function and η is a given function.

Third type perturbation is the perforation of the domain along a curve. We choose a curve inside the strip assuming that this curve is either infinite or finite and closed. Along this curve, we cut out small holes close to each other. The distribution and the sizes of the holes are non-periodic. We impose the Dirichlét or the Robin condition on the holes, assuming that different boundary condition can be imposed on different holes.

For all cases described above, we impose the Dirichlét condition on the rest of the boundary. The operator is assumed to be self-adjoint.

We describe the homogenized operators subject to the structure of the perturbation. The main result is the proof of the uniform resolvent convergence of the perturbed operator to the homogenized one and the estimates for the rates of convergence.

This work is based on a series of joint papers with G. Cardone, R. Bunoiu, T. Durante, L. Faella, and C. Perugia.

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On Solvability of Periodic Problem for Second-Order Functional Differential Equations

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Sharp estimates for the unique solvability of the periodic boundary value problem for linear second-order functional differential equations are obtained. Conditions for solvability in the form of integral restrictions on functional operators can be found in works by I. Kiguradze, R. Hakl, A. Lomtatidze, S. Mukhigulashvili, A. Ronto, J. Šremr, and others. Here we determine the best constants for the point-wise restrictions.

Consider the periodic boundary value problem

$$\begin{cases} \ddot{x}(t) = \lambda(Tx)(t) + f(t) \quad \text{for almost all } t \in [a, b], \\ x(a) = x(b), \quad \dot{x}(a) = \dot{x}(b), \end{cases} \tag{1}$$

where λ is a real number, $T : \mathbf{C}[a, b] \to \mathbf{L}[a, b]$ is a bounded linear operator, $f \in \mathbf{L}[a, b]$, and a solution $x : [a, b] \to \mathbf{R}$ has an absolutely continuous derivative.

Given a function $p \in \mathbf{L}[a, b]$, define the functions

$$q_{t_1,t_2}(t) \equiv \begin{cases} \frac{(t-a)(t_2-t_1)}{b-a}, \quad t \in [a, t_1), \\ t_2 - t - \frac{(b-t)(t_2-t_1)}{b-a}, \quad t \in [t_1, t_2), \\ -\frac{(b-t)(t_2-t_1)}{b-a}, \quad t \in [t_2, b], \end{cases}$$

$$q_{t_1,t_2,p}(t) \equiv q_{t_1,t_2}(t) - \int_a^b p(s) q_{t_1,t_2}(s)\, ds, \quad t \in [a, b].$$

For every $z : [a, b] \to \mathbf{R}$, denote $z^+(t) \equiv (z(t) + |z(t)|)/2$, $z^-(t) \equiv (z(t) - |z(t)|)/2$.

Theorem. *Let $T\mathbf{1} = p$, $\int_a^b p(t)\,dt = 1$, the functionals $x \mapsto (Tx)(t)$ be monotone for almost all $t \in [a, b]$, and*

$$\lambda \neq 0, \quad |\lambda| < \frac{1}{\max\limits_{a \leqslant t_1 < t_2 \leqslant b} \int_a^b \left(p^+(t) q^+_{t_1,t_2,p}(t) + p^-(t) q^-_{t_1,t_2,p}(t)\right)\,dt}. \tag{2}$$

Then periodic problem (1) *has a unique solution.*

The constant on the right-hand side in (2) is exact and cannot be increased.

Corollary. *Let $r \in \mathbf{L}[a, b]$ with $R \equiv \int_a^b r(t)\,ds > 0$ be given. The periodic problem*

$$\begin{cases} \ddot{x}(t) = r(t)x(h(t)) + f(t) & \text{for almost all } t \in [a, b], \\ x(a) = x(b), \quad \dot{x}(a) = \dot{x}(b) \end{cases}$$

has a unique solution for every measurable function $h : [a, b] \to [a, b]$ if and only if

$$\max_{a \leqslant t_1 < t_2 \leqslant b} \int_a^b \left(r^+(t) q^+_{t_1,t_2,r/R}(t) + r^-(t) q^-_{t_1,t_2,r/R}(t)\right)\,dt < 1.$$

The best constants in solvability conditions (2) for some functions p are obtained in the explicit form.

Differential-Algebraic Solutions of the Heat Equation

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We discuss solutions of the heat equation $\frac{\partial}{\partial t}\psi(z, t) = \frac{1}{2}\frac{\partial^2}{\partial z^2}\psi(z, t)$ in the ansatz

$$\psi(z, t) = f(t) \exp\left(-\frac{1}{2}h(t)z^2\right) \Phi(z, t)$$

with additional conditions on $\Phi(z, t)$ that reduce the heat equation to a homogeneous non-linear ordinary differential equation. The corresponding Burgers equation solutions are obtained via the Cole-Hopf transform.

In our ansatz we have the following classical examples of the heat equation solutions: the flat wave solution with $h(t) = 0$ and $\Phi(z, t) = \Phi(z, 0)$; the Gaussian normal distribution with the standard deviation $\sigma = \sqrt{2at}$, where $f(t) = (2\pi\sigma^2)^{-1/2}$, $h(t) = \sigma^{-2}$, and $\Phi(z, t) = 1$; the solution in terms of the elliptic theta-function described in the ansatz

$$\theta_1(z, t) = \sqrt{\frac{\omega}{\pi}} \sqrt[8]{\Delta} \exp(-2\omega\eta z^2) \sigma(\hat{z}, g_2, g_3),$$

where $\sigma(\hat{z}, g_2, g_3)$ is the Weierstrass sigma-function, $\omega t = \omega'$, and $\hat{z} = 2\omega z$. In the last case, our method gives (see [1]) the Chazy-3 equation

$$y'''(t) = 2y(t)y''(t) - 3y'(t)^2.$$

By a *differential-algebraic solution of the heat equation*, we mean a solution in our ansatz, satisfying the additional conditions that $\psi(z,t)$ is regular for $z=0$ and $\Phi(z,t)$ or $\Phi'(z,t)$ is an even function in z such that the series decomposition coefficients $\Phi_k(t)$ of z^{2k} are homogeneous polynomials of degree $-2k$ in $x_2,\dots,x_k$, where $\deg x_q = -2q$, $q = 2,3,\dots$. A differential-algebraic solution is called an *n-ansatz solution of the heat equation* if $\Phi_k(t)$ are homogeneous polynomials of n variables $x_2(t),\dots,x_{n+1}(t)$.

Consider the differential operator

$$\mathcal{L} = \frac{\partial}{\partial y_1} - \sum_{s=1}^{\infty}(s+1)sy_s\frac{\partial}{\partial y_{s+1}}.$$

A polynomial $P(y_1,\dots,y_{n+2})$ is called admissible if it is a homogeneous polynomial with respect to the grading $\deg y_k = -2k$ and $\mathcal{L}P(y_1,\dots,y_{n+2}) = 0$.

We prove that a differential-algebraic solution of the heat equation is an n-ansatz solution if and only if the function $h = h(t)$ is a solution of the ordinary differential equation $P(h,h',\dots,h^{(n+1)}) = 0$ with admissible $P(y_1,\dots,y_{n+2})$. Fixed such a function $h(t)$, we find an expression for $f(t)$ in terms of $h(t)$ and recurrent expressions for $\Phi_k(t)$, $k = 2,3\dots$, as differential polynomials of $h(t)$, see [2].

Examples of ordinary differential equations obtained from admissible polynomials for small n are

$$h' = -h^2, \quad h'' = -6hh' - 4h^3, \quad h''' = -12hh'' + 18h'^2 + c_3(h'+h^2)^2,$$
$$h'''' = -20hh''' + 24h'h'' - 96h^2h'' + 144hh'^2 + c_4(h'+h^2)(h''+6hh'+4h^3),$$

where c_3 and c_4 are constants.

As $c_3 = 0$, the third-order equation becomes the Chazy-3 equation after the substitution $y(t) = -6h(t)$. The values of c_3 for which this equation has the Painléve property were described in classical papers.

The fourth-order equation has the Painléve property only in the case where its general solution is rational (see [3]).

It is shown in [3] that the next (fifth-order) equation has series of parameters satisfying the Painléve test.

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On Expansions of Differential Operators in Banach Spaces

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It is well-known that the usual theory of expansions of partial differential operators (Vishik, Hörmander, Berezansky, Dezin) or, equivalently, the general theory of boundary value problems is built in the Hilbert space $L_2(\Omega)$. In this report, we bring first results of the corresponding theory built for Banach spaces.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We consider expansions of an operator $\mathcal{L}^+ = \sum_{|\alpha|\leqslant l} a_\alpha(x) D^\alpha$, $D^\alpha = \frac{(-i\partial)^{|\alpha|}}{\partial x^\alpha}$ (initially acting in the space $C^\infty(\Omega)$) and the formal adjoint operator $\mathcal{L}^+\cdot = \sum_{|\alpha|\leqslant l} D^\alpha(a^*_\alpha(x)\ \cdot)$, where $a_\alpha(x)$ are $N \times N^+$ matrices with entries $(a_\alpha)_{ij} \in C^\infty(\bar{\Omega})$, $a^*_\alpha(x)$ is the adjoint matrix.

For $p > 1$ and $q = p/(p-1)$, we introduce the graph norms $\|u\|_{L,p} = \|u\|_{L_p(\Omega)} + \|\mathcal{L}u\|_{L_p(\Omega)}$, $\|u\|_{L,q}$, $\|u\|_{L^+,p}$, and $\|u\|_{L^+,q}$. Then we build the minimal operators L_{p0}, L_{q0}, L^+_{p0}, and L^+_{q0} whose domains are closures of $C_0^\infty(\Omega)$ in the corresponding graph norms, and the maximal operators $L_p := (L^+_{q0})^*$, $L_q := (L^+_{p0})^*$, L^+_p, and L^+_q. Any operator $L_{pB} = L_p|_{D(L_{pB})}$ such that $D(L_{p0}) \subset D(L_{pB}) \subset D(L_p)$ is called an expansion (of L_{p0}), and the expansion $L_{pB} : D(L_{pB}) \to [L_p(\Omega)]^{N^+} =: B^+_p$ is called *solvable* if there exists a continuous two-sided inverse operator $L^{-1}_{pB} : B^+_p \to D(L_{pB})$, $L_{pB}L^{-1}_{pB} = \mathrm{id}_{B^+_p}$, $L^{-1}_{pB}L_{pB} = \mathrm{id}_{D(L_{pB})}$.

Here as usual one introduces the notion of the boundary value problem in the form $L_p u = f$, $\Gamma u \in B$, where a subspace B of the space $C(L_p) =: D(L_p)/D(L_{p0})$ defines the homogenous boundary value problem similar to the Hörmander definition, $\Gamma : D(L_p) \to C(L_p)$ being the factor-mapping. The two Vishik conditions in the Hilbert case turn to the following four conditions in the Banach case: the operator L_{p0} has a bounded left inverse (condition (1_p)) and the same is about the operators L_{q0} (condition (1_q)), L^+_{p0} (condition (1^+_p)), and L^+_{q0} (condition (1^+_q)). Then we prove the theorems.

Theorem 1. *The operator L_{p0} has a solvable expansion iff the conditions (1_p) and (1^+_q) are fulfilled.*

Theorem 2. *Under conditions (1_p) and (1^+_p), we have decomposition $D(L_p) = D(L_{p0}) \oplus \ker L_p \oplus W_p$, where W_p is a subspace of $D(L_p)$ such that $L_p|_{W_p} : W_p \to \ker L^+_p$ is an isomorphism.*

Theorem 3. *Under conditions (1_p) and (1^+_p), any solvable expansion L_{pB} can be decomposed into the direct sum $L_{pB} = L_{p0} \oplus L^\partial_{pB}$, where $L^\partial_{pB} : B \to \ker L^{-1}_{p0}$ is an isomorphism.*

Theorem 4. *Under conditions (1_p) and (1^+_p), any linear subspace $B \subset C(L_p)$ with the properties* 1) *$\Gamma^{-1}_p B \cap \ker L_p = 0$ and* 2) *there exists an operator $M_p : \ker L^{-1}_{p0} \to D(L_p)$ such that* a) *$L_p M_p = id|_{\ker L^{-1}_{p0}}$ and* b) *$\operatorname{Im} M_p \subset \Gamma^{-1}_p B$, generates a well-posed boundary value problem, i.e. a solvable expansion L_{pB} with domain $D(L_{pB}) = \Gamma^{-1}B$.*

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Approximation of Feller Semigroups via Feynman Formulae

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We consider evolution semigroups generated by some differential and integro-differential (hence non-local) operators associated with Feller stochastic processes in $\mathbb{R}^d$. This class of operators includes second order elliptic operators with bounded continuous coefficients, fractional Laplacians, relativistic Hamiltonians, and many others. We represent the considered semigroups as limits of iterated n-fold integrals of elementary functions as n tends to infinity. Such representations of evolution semigroups are called *Feynman formulae*. They allow one to approximate solutions of corresponding evolution equations and, consequently, to model the dynamics numerically. Recently the method of Feynman formulae has been applied to represent solutions of different classes of evolution equations on various geometrical structures.

For such semigroups we present two types of Feynman formulae: *Lagrangian* and *Hamiltonian* ones. The limits of iterated integrals in Lagrangian Feynman formulae coincide with functional integrals (i.e. integrals over infinite dimensional path spaces) with respect to probability measures. Such integrals are usually called Feynman–Kac formulae. Representation of solutions of evolution equations by Feynman–Kac formulae allows one to investigate the corresponding dynamics by means of stochastic analysis, e.g., by the method of Monte–Carlo. The limits of iterated integrals in Hamiltonian Feynman formulae coincide with functional integrals with respect to Feynman type pseudomeasures. Such integrals are usually called Feynman path integrals and are an important tool in quantum mechanics. Therefore, the apparatus of Feynman formulae allows one to establish Feynman–Kac formulas, to define rigorously several Feynman path integrals over paths both in configuration and in phase space, to calculate some Feynman path integrals, and also to connect some Feynman path integrals with functional integrals with respect to probability measures.

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A Few New Results on Fully Nonlinear Degenerate Elliptic Equations

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In the first part of the talk we consider fully nonlinear degenerate elliptic equations with zero and first order terms. We provide a priori upper bounds and characterize the existence of entire subsolutions for such equations under growth conditions on the lower order coefficients which extend the classical Keller–Osserman condition for semilinear equations.

In the second part we characterize the validity of the Maximum Principle in bounded domains for fully nonlinear degenerate elliptic operators in terms of the sign of a generalized principal eigenvalue. The introduction of a new notion of principal eigenvalue is required in our framework because of the degeneracy of the operators involved.

Some New Results for Approximate and Near Weak Invariance with Respect to Nonautonomous Differential Inclusions in Banach Spaces

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Let X be a real Banach space, $I \subset \mathbb{R}$ a nonempty bounded interval and $F : I \times X \rightsquigarrow X$ a given multi-function. We recall that an exact solution to the Cauchy problem

$$\begin{cases} x'(t) \in F(t, x(t)), \\ x(\tau) = \xi \end{cases} \tag{1}$$

on $[\tau, T] \subset I$ is a function $x \in W^{1,1}([\tau, T]; X)$ which satisfies (1) a.e. on $[\tau, T]$. We study here approximate and near weak invariance of a cylindrical subset $\mathcal{K} = I \times K$, where $K \subset X$, with respect to the multi-function F by means of an appropriate tangency concept and the Lipschitz conditions on F. For a given $\varepsilon > 0$, we introduce the notion of an ε-solution as follows: by an ε-solution to (1) we mean a function $x \in W^{1,1}([\tau, T]; X)$ solving the associate Cauchy problem

$$\begin{cases} x'(t) \in F(t, x(t) + \varepsilon B), \\ x(\tau) = \xi \end{cases}$$

on $[\tau, T]$. Here B denotes the closed unit ball in X. The classical concept of weak invariance requires that for each initial data (τ, ξ) in $\mathcal{K}$ there exists an exact solution to Cauchy problem (1) which remains in $\mathcal{K}$ at least for a short time. The concept of approximate and near weak invariance of a set $\mathcal{K}$ generalizes the classical one. More exactly, we say that the set $\mathcal{K}$ is approximate (near) weakly invariant with respect to F if for any $(\tau, \xi) \in \mathcal{K}$, there exists $T > \tau$ such that for any $\varepsilon > 0$, there exists an $\varepsilon-$ solution (an exact solution) to (1) satisfying

$$dist(x(t); K) \leqslant \varepsilon, \forall t \in [\tau, T].$$

Existence and Uniqueness Theorems for the Full Three-Dimensional Ericksen–Leslie System

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We consider the Ericksen–Leslie system of equations modelling hydrodynamics of nematic liquid crystals,

$$\begin{cases} \dot{\mathbf{u}} - \mu\Delta\mathbf{u} = -\nabla p - \frac{\partial}{\partial x_j}\left(\frac{\partial \mathcal{F}}{\partial \mathbf{n}_{x_j}} \cdot \nabla\mathbf{n}\right) + \mathbf{F}, \quad \mathrm{div}\mathbf{u} = 0, \\ J\ddot{\mathbf{n}} - 2q\mathbf{n} + \mathbf{h} = \mathbf{G}, \quad \|\mathbf{n}\| = 1, \end{cases} \tag{1}$$

where

$$\mathcal{F}(\mathbf{n}, \nabla\mathbf{n}) := \frac{1}{2}\Big(K_1(\mathrm{div}\mathbf{n})^2 + K_2(\mathbf{n}\cdot\mathrm{curl}\mathbf{n})^2 + K_2\|\mathbf{n}\times\mathrm{curl}\mathbf{n}\|^2\Big),$$

$$\mathbf{h} := \frac{\partial \mathcal{F}}{\partial \mathbf{n}} - \frac{\partial}{\partial x_j}\left(\frac{\partial \mathcal{F}}{\partial \mathbf{n}_{x_j}}\right).$$

Here $\mathbf{u}$ and $\mathbf{n}$ are unknown vector fields, while p and q are unknown scalar functions. The constants μ, J, K_1, and K_2, and the vector fields F are G are given.

Theorem 1 (Periodic solution of the Ericksen–Leslie system)**.** *Let $\mathbb{T}$ be the three-dimensional flat torus $\mathbb{R}^3/\mathbb{Z}^3$. Suppose $\mathbf{u}(x,0) \in Sol_2^2(\mathbb{T})$, where Sol_2^2 consists of all solenoidal vector fields from W_2^2, $\mathbf{n}(x,0)\times\dot{\mathbf{n}}(x,0) \in W_2^2(\mathbb{T})$, $\mathbf{n}(x,0) \in W_2^3(\mathbb{T})$, and $\mathbf{F} \in L_2((0,T); W_2^1(\mathbb{T}))$, $\mathbf{G} \in L_1((0,T); W_2^2(\mathbb{T}))$.*

Then for some $0 < T_0 < T$ there exists a quadruple $(\mathbf{u}, \mathbf{n}, p, q)$ satisfying (1) *for almost all $(x,t) \in Q_{T_0} = \mathbb{T}\times(0,T_0)$ and the initial conditions. Under some special assumptions, this solution is unique.*

Theorem 2 (Solution in the bounded domain)**.** *Assume that Ω is a C^2-domain. Let $\mathbf{n}(x,0) - \mathbf{n}_0 \in \overset{\circ}{W}{}_2^3(\Omega)$ for some $\mathbf{n}_0 \equiv const$, $\mathbf{n}(x,0)\times\dot{\mathbf{n}}(x,0) \in W_2^2(\Omega)$, $\mathbf{u}(x,0) \in \overset{\circ}{Sol}{}_2^1(\Omega) \cap W_2^2(\Omega)$, $\Delta\mathbf{u}(x,0)\big|_{\partial\Omega} = 0$, and for some $d > 0$ we have $\mathbf{n}(x,0) = \mathbf{n}_0$, $\mathbf{n}(x,0)\times\dot{\mathbf{n}}(x,0) = 0$ if $dist(x,\partial\Omega) < d$.*

Suppose also that $\mathbf{F} \in L_2((0,T); W_2^1(\Omega))$, $\mathbf{G} \in L_1((0,T); W_2^2(\Omega))$, and $\mathbf{G}$ is equal to zero in a d-neighborhood of $\partial\Omega$. Then there exists a strong solution $(\mathbf{u}, \mathbf{n}, p, q)$ to (1) *for almost all $(x,t) \in \Omega\times(0,T_0)$, $T_0 \leqslant T$, and satisfying the initial and the boundary conditions. Under some special assumptions, this solution is unique.*

Reaction Diffusion Equations with Hysteresis in Higher Spatial Dimensions

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Reaction diffusion equations with a hysteretic discontinuity were first introduced to model bacteria growth involving one diffusing (Salmonella typhimurium) and two non-diffusing (nutrient and buffer) substances. At each spatial point, the reaction term can be in one of two configurations where the diffusing substances determine two distinct thresholds for the switching behavior. The existence of distinct thresholds means the current configuration is determined by the past concentrations at that point. The dependence on past history and the existence of sub-domains in different configurations are the defining features of the problem. The interplay of the diffusion and hysteresis is responsible for the evolution of free boundaries separating these sub-domains as well as the appearance of spatio-temporal patterns. Numerical analysis of the problem on the unit disk qualitatively reproduced the concentric ring pattern observed in experiments, however rigorous results only exist in one spatial dimension. Results in higher dimensions must account for free boundaries with far more complicated topologies. In this talk we present analytic results in higher dimensions where the free boundaries are non intersecting and the data crosses these boundaries transversely.

Dissipative Hölder Solutions to the Incompressible Euler Equations

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We consider local in time solutions to the Cauchy problem for the incompressible Euler equations on the 3-dimensional torus which are continuous or Hölder continuous for any exponent $\theta < 1/16$. Using the techniques introduced by De Lellis and Szekelyhidi in 2012, we prove that for any given positive and smooth function of time e, there exist infinitely many (Hölder) continuous initial vector fields starting from which there exist infinitely many (Hölder) continuous solutions with total kinetic energy e.

A Priori Estimate of Solutions for a Model Nonlocal Problem

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We consider an ordinary differential operator of even order with nonlocal boundary conditions. Our aim is to obtain an a priori estimate for the solutions. In this work, we accomplish the task for a model problem with a 4-th order operator and integral conditions.

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Optimal Stationary Cyclic Utilization of Renewable Resources[1]

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We search an optimal stationary mode of utilization of spatially distributed renewable resource. The objective function is the maximum averaged profit, where the average is taken over the time interval or the efforts used to extract the resource.

In some cases, problems of this type are analogous to those ones from search theory [1], for example, when the resource recovers after the harvesting totally by the next cycle [2]. On the other hand, the problems lead to new interesting mathematical tasks if the resource has some nonlinear law of recovering [3–5].

The talk is devoted to the recent results and observations done in the analysis of stationary modes of utilization of spatially distributed renewable resource. For example, in the optimal mode of utilization, stops can appear between consequent cycles if the recovering is not "too fast" [5].

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Variations on the Bombieri–de Giorgi–Giusti Minimal Graph

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Bombieri, De Giorgi, and Giusti discovered a nontrivial entire solution to the minimal surface equation in $\mathbb{R}^8$. Analysis of its asymptotics and its Jacobi operator leads to interesting examples in some classical elliptic PDE questions for large space dimensions.

Maximal Regular Boundary Value Problems for Second Order Abstract Elliptic Differential Equations in UMD Banach Spaces

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In this talk, we consider a class of boundary value problems for abstract second order elliptic differential equations in UMD Banach spaces. The boundary conditions contain a spectral parameter. Several conditions are obtained that guarantee the maximal regularity, Fredholmness, estimates for the resolvent, and the completeness of the root elements of the considered problem. These results are then applied to the study of a class of nonlocal boundary value problems for regular elliptic partial differential equations in cylindrical domains.

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Inverse Problem for Quasi Linear System of Partial Differential Equations with Nonlocal Boundary Condition Containing Delay Argument[1]

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We consider the following problem:

$$u_x + a_t = 0, \quad 0 \leqslant x \leqslant l, \quad 0 \leqslant t \leqslant T, \tag{1}$$

$$a_t = u - \psi(a), \quad 0 \leqslant x \leqslant l, \quad 0 \leqslant t \leqslant T, \tag{2}$$

$$u(0,t) = \mu(t) + u(l, \alpha(t)), \quad 0 \leqslant t \leqslant T, \tag{3}$$

$$a(x,0) = 0, \quad 0 \leqslant x \leqslant l, \tag{4}$$

where the function $\alpha(t)$ satisfies the condition $0 \leqslant \alpha(t) < t$, $t \in (0,T]$.

Problem (1)–(4) can be treated as a mathematical model of a filtration system. In this system, a gas or a liquid passed through a filter of length l is fed again to the input of the system with some time delay specified by the function $\alpha(t)$.

We assume that the functions $\psi(s)$, $\mu(t)$, and $\alpha(t)$ satisfy the following conditions:

$$\psi \in C^2(R), \quad \psi(0) = 0, \quad 0 < \psi'(s) \leqslant \psi_1, \quad |\psi''(s)| \leqslant \psi_2, \quad s \in R; \tag{5}$$

$$\mu \in C^3[0,T], \quad \mu(0) = 0, \quad \mu'(t) > 0, \quad t \in [0,T]; \tag{6}$$

$$\alpha \in C^3[0,T], \quad \alpha(0) = 0, \quad \alpha'(0) < 1, \quad 0 \leqslant \alpha'(t) \leqslant 1, \quad t \in [0,T]. \tag{7}$$

Theorem 1. *If conditions* (5)–(7) *hold and* $\psi \in C^3[0, \psi^{-1}(2\mu(T))]$*, then there exists a unique pair of functions* $u(x,t), a(x,t) \in C^3\{[0,l] \times [0,T]\}$ *satisfying* (1)–(4).

We consider the following inverse problem. Suppose that the functions $\mu(t)$, $\alpha(t)$, and

$$u(x_0, t) = g(t), \quad 0 \leqslant t \leqslant T, \tag{8}$$

are given, while the functions $\psi(s)$, $u(x,t)$, and $a(x,t)$ solving (1)–(4) and (8) are to be determined.

Definition. A triple $\{\psi(s), u(x,t), a(x,t)\}$ is called a solution to inverse problem (1)–(4) and (8) if $\psi(s)$ satisfies conditions (5), $\psi \in C^3[0, \psi^{-1}(2\mu(T))]$, $u, a \in C^3[Q_T]$, and $\psi(a(x,t)), u(x,t), a(x,t)$ solve (1)–(4) and (8).

Theorem 2. *Let conditions* (6) *and* (7) *be satisfied and*

$$e^{-2x_0} + (\alpha'(0))^6[3e^{-(l+2x_0)} + e^{-3l}] < 1.$$

If $\{\psi_i(s), u_i(x,t), a_i(x,t)\}$*,* $i = 1, 2$*, are solutions of inverse problem* (1)–(4) *and* (8)*, then* $u_1(x,t) = u_2(x,t)$*,* $a_1(x,t) = a_2(x,t)$ *for* $x \in [0,l]$*,* $t \in [0,T]$*, and* $\psi_1(s) = \psi_2(s)$ *for* $s \in [0, a_1(0,T)]$.

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Fixed Points Techniques and Stability in Totally Nonlinear Neutral Differential Equations with Functional Delay

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In this work we present the fixed point method to prove asymptotic stability results of the zero solution for a class of totally nonlinear differential equations with functional delay. Such a problem has proved very challenging in the theory of Liapunov's direct method. However, we show that a modification of the Krasnoselskii theorem fits very nicely so that asymptotic stability is readily concluded.

Reversibility States of Linear Differential Operators with Periodic Unbounded Operator Coefficients

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Let X be a Banach space over the field $\mathbb{C}$. Let $\operatorname{End} X$ be the Banach algebra of all endomorphisms of the space X. We consider the following function spaces. By $C = C(\mathbb{R}, X)$ we denote the Banach space of all continuous functions on $\mathbb{R}$, and the subspace $C_w(\mathbb{R}, X)$ consists of all w-periodic functions. By $L^p = L^p(\mathbb{R}, X)$, $1 \leqslant p \leqslant \infty$, we denote the space of all Bochner-measurable p-integrable functions, and $L^p_w = L^p_w(\mathbb{R}, X)$ is the space of all Bochner-measurable locally p-integrable w-periodic functions. Let $\mathcal{F} = \mathcal{F}(\mathbb{R}, X)$ and $\mathcal{F}_w = \mathcal{F}_w(\mathbb{R}, X)$ stand for one of the above introduced function spaces ($\mathcal{F} = C$ or $\mathcal{F} = L^p$). We also introduce the corresponding spaces of two-sided sequences. We denote by $\mathcal{F}_d = \mathcal{F}_d(\mathbb{Z}, X)$ the space of two-sided sequences, coinciding with the space of bounded sequences if $\mathcal{F} \in \{C, L^\infty\}$, with the space of p-integrable sequences if $\mathcal{F}$ is L^p, $1 \leqslant p < \infty$, and with the space of constant sequences if $\mathcal{F}$ is the space of w-periodic functions.

The symbol Δ denotes the set $\{(t, s) \in \mathbb{R}^2 : s \leqslant t\}$. A map $\mathcal{U} : \Delta \to \operatorname{End} X$ is called a (*strongly continuous*) *w-periodic family of evolution operators* on $\mathbb{R}$ if the follow conditions are fulfilled:

1. $\mathcal{U}(t, t) = I$ is the identity operator for all $t \in \mathbb{R}$;
2. $\mathcal{U}(t, s)\mathcal{U}(s, \tau) = \mathcal{U}(t, \tau)$ for all $\tau \leqslant s \leqslant t$;
3. the map $(t, s) \longmapsto \mathcal{U}(t, s)x : \Delta \to X$ is continuous for all $x \in X$;
4. $\mathcal{U}(t + w, s + w) = \mathcal{U}(t, s)$ for all $(t, s) \in \Delta$.

We define an operator $\mathcal{L} : D(\mathcal{L}) \subset \mathcal{F} \to \mathcal{F}$ as follows. A function $x \in \mathcal{F}$ belongs to $D(\mathcal{L})$ if there exists a function $f \in \mathcal{F}$ such that

$$x(t) = \mathcal{U}(t, s)x(s) - \int\limits_s^t \mathcal{U}(t, \tau)f(\tau)d\tau, \quad s \leqslant t.$$

In this case, we set $\mathcal{L}x = f$.

Definition 1. Let $A : D(A) \subset Y \to Y$ be a closed linear operator acting in a Banach space Y, and let operator A satisfy at least one of the following conditions:

1. $\operatorname{Ker} A = \{0\}$ (A is injective);
2. $1 \leqslant n = \dim \operatorname{Ker} A < \infty$;
3. $\dim \operatorname{Ker} A = \infty$;
4. $\operatorname{Ker} A$ is a complementable subspace in $D(A)$ (with respect to the graph norm) or in the space Y;
5. $\operatorname{Im} A = \overline{\operatorname{Im} A}$;
6. $\operatorname{Im} A$ is a closed complementable subspace in the space Y;
7. $\operatorname{Im} A$ is a closed complementable subspace in the space Y and $\operatorname{codim} \operatorname{Im} A = n < \infty$;
8. $\operatorname{Im} A = Y$ (A is surjective operator)
9. $\overline{\operatorname{Im} A} \neq Y$;
10. $A^{-1} \in \operatorname{End} X$.

If all the conditions from a set of conditions $S = \{i_1, \dots, i_k\}$ are fulfilled for A, then we say that the operator A is in the invertibility state S. The set of all invertibility states of an operator A is denoted by $\mathrm{St}_{\mathrm{inv}}(A)$.

Introduce a difference operator $\mathcal{D} : \mathcal{F}_d \to \mathcal{F}_d$ given by the formula

$$(\mathcal{D}x_d)(n) = x_d(n) - \mathcal{U}(w,0)x_d(n-1), \quad x_d \in \mathcal{F}_d,\ n \in \mathbb{Z}.$$

Theorem 1. *The operator $\mathcal{L}$ and the difference operator $\mathcal{D}$ have the same set of invertibility states,*

$$\mathrm{St}_{\mathrm{inv}}(\mathcal{L}) = \mathrm{St}_{\mathrm{inv}}(\mathcal{D}).$$

Functions of Noncommuting Operators, Adiabatic Approximation in Homogenization Problems, and Applications

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As a rule, homogenization methods are used for constructing asymptotic solutions such that their leading terms are sufficiently smooth functions. There exists a great number of publications devoted to homogenization, among them are the well-known monographs by N. Bakhvalov and G. Panasenko, E. Khruslov and V. Marchenko,

V. Zhikov, S. Kozlov and O. Oleinik, A. Bensoussan, J.-L. Lions, G. Papanicolaou, S. Nazarov, etc. We show that homogenization of many linear operators with oscillating coefficients could be done in the frame of the adiabatic approximation based on pseudodifferential operators (functions of noncommuting operators) and the Maslov operator methods. To this end, we first discuss the definition of a function of noncommuting operators (pseudodifferential operators with parameter) based on the Feynmann–Maslov ordering, and their properties like composition, etc. Then we explain how this type of pseudodifferential operators could be used in many adiabatic problems and homogenization theory. This approach allows one to reproduce the well-known homogenization results in another way but also taking into account the so-called dispersion effects leading to a change in the structure of the original equation. We apply this theory to derive the 2-D equation describing water waves over an oscillating bottom, and also graphene ribbon. As an example, we discuss the asymptotics of the solution to the Cauchy problem with localized and rapidly oscillating initial data.

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Submanifolds of Family of Periodic Eigenvalue Problems

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We consider the family

$$-y'' + p(x)y = \lambda y, \quad y(0) - y(2\pi) = y'(0) - y'(2\pi) = 0$$

of periodic eigenvalue problems with 2π-periodic real potential $p \in C^0(2\pi)$ as a functional parameter. For any potential p, the spectrum of the problem consists of isolated real eigenvalues, the multiplicity of an eigenvalues does not exceed two, and the spectrum has the form

$$\lambda_0(p) < \lambda_1^-(p) \leqslant \lambda_1^+(p) < \ldots < \lambda_k^-(p) \leqslant \lambda_k^+(p) < \ldots \to \infty.$$

Let $\Delta\lambda \geqslant 0$ be a constant number. We consider the subset

$$P_k(\Delta\lambda) := \{p \mid \lambda_k^+(p) - \lambda_k^-(p) = \Delta\lambda\} \subset C^0(2\pi) \quad k = 1, 2, \ldots.$$

If $\Delta\lambda = 0$, then we have Arnold's subset of potentials with a double k-eigenvalue [1]. We provide a novel description of topological structure of the subsets $P_k(\Delta\lambda)$ (another approach to this problem is described in [2]). In particular, we prove Arnold's "hypothesis of transversality" that the subset $P_k(0) \subset C^0(2\pi)$ is a smooth submanifold of codimension two.

Then we find the linking coefficient of the submanifold $P_k(0)$ and the loop of shifts $l(k) \subset C^0(2\pi)$ to be $2k$, where

$$l(k) := \{p(x+t) \notin P_k(0) \mid t \in [0, 2\pi]\}.$$

We also find the linking coefficient of the submanifold $P_k(0)$ and the loop of periodic stationary solutions of the Korteweg-de Vries equations.

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On the Complexity of Skew Products of Maps of an Interval

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We give here the observation of author's results about the complexity of C^1-smooth skew products of maps of an interval. We focus on the space of C^1-smooth skew products of maps of an interval, containing maps with the discontinuous Ω-function and a countable set of suitable discontinuous functions for the Ω-function (see [1]).

In particular, we distinguish the subspace $\widetilde{T}^1_*$ of the above space, consisting of maps F such that the set of discontinuity points of the Ω-function of $F \in \widetilde{T}^1_*$ is an infinite ω-limit set containing a periodic point of the quotient map f; the family of fiber maps over the continuity points of the Ω-function of $F \in \widetilde{T}^1_*$ is stable in general.

We construct an example of the skew product from $\widetilde{T}^1_*$ and prove that skew products from $\widetilde{T}^1_*$ with an admissible depth of the center (i.e., not higher than the second class), exceeding any given transfinite ordinal of the second class, are everywhere dense in $\widetilde{T}^1_*$.

These results demonstrate impossibility of complete dynamical description of maps of the space $\widetilde{T}^1_*$ based on the concept of the Ω-conjugacy.

We also formulate some unsolved problems.

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Modified Closed Boundary Conditions for Impulsive Fractional Differential Equations

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Boundary-value problems for fractional differential equations with impulse effects have recieved great attention in the related literature. Especially, Ahmad et al. [1] considered closed boundary conditions for fractional differential equations. What is more, Ergören and Kiliçman [2] and Wang et al. [3] did this for impulsive fractional differential equations. In the present study, we establish some existence results concerning boundary-value problems for impulsive fractional differential equations with *modified* closed boundary conditions.

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On Properties of Solutions to Kawahara Equation

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The Kawahara equation

$$u_t - u_{xxxxx} + bu_{xxx} + au_x + uu_x = 0$$

is sometimes called the Korteweg–de Vries equation of 5-th order. It describes propagation of nonlinear waves in weakly dispersive media. Consider the initial-boundary value problem in a half-strip $\Pi_T^+ = (0,T) \times \mathbb{R}_+$ for an arbitrary $T > 0$ with initial and boundary data

$$u(0,x) = u_0(x), \qquad u(t,0) = u_x(t,0) = 0.$$

Let $u_0 \in L^{\alpha}_{2,+} = \{u : (1+x)^{\alpha} u \in L_2(\mathbb{R}_+)\}$ for a certain positive α. It was then proved by K. Sangare and A. V. Faminskii the existence of a weak solution to this problem in the space

$$u \in C_w([0,T]; L^{\alpha}_{2,+}), \qquad u_{xx} \in L_2(0,T; L^{\alpha-1/2}_{2,+}),$$

and its uniqueness under the additional condition $\alpha \geqslant 3/8$. Now we consider the properties of internal smoothness of these solutions.

Theorem 1. *Suppose $u_0 \in L^{\alpha}_{2,+}$ with $\alpha \geqslant 1/2$ and m is a natural number such that $m \leqslant 4\alpha$. Then one has*

$$(1 + x - x_0)^{\alpha - m/4} \partial^m_x u \in C_w([\delta, T]; L_2(x_0; +\infty))$$

for any $\delta \in (0,T)$ and $x_0 > 0$ (if $m \leqslant 2$, then one can take $x_0 = 0$). Moreover, $\partial^{m+2}_x u \in L^{loc}_2(\Pi^+_T)$.

Some results on the existence of continuous derivatives and their estimates in Hölder spaces are also established.

This whole theory can be extended to more general equations, for example, those ones containing an additional term $g(x)u$ on the left-hand side. Moreover, if a non-negative function $g \in L_\infty(\mathbb{R}_+)$ is strictly positive at $+\infty$, then the following large-time decay property is established:

$$\|u(t,\cdot)\|_{L_2(\mathbb{R}_+)} \leqslant ce^{c_0 t} \quad (\forall\ t \geqslant 0)$$

uniformly with respect to $\|u_0\|_{L_2(\mathbb{R}_+)}$ with some positive constants c and c_0.

Perturbation Methods for Inverse Problems Related to Degenerate Differential Equations

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Provided the single-valued perturbing operators fulfill some suitable interpolation property, we first derive perturbation theorems for abstract multivalued linear operators satisfying a resolvent condition of weak parabolic type. We then apply our theorems to inverse problems on degenerate differential equations, supplying a new perturbation method which avoids any fixed-point argument and essentially consists in reducing the original inverse problem in an auxiliary direct one.

Classification of Solutions for Elliptic PDEs

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Classifying solutions of PDEs has been a very interesting topic. We begin by various celebrated classification results for solutions of elliptic PDEs such as Lane–Emden conjecture and De Giorgi's conjecture. These conjectures have attracted many experts in the field for a few decades. Later in this talk, we state counterparts of these conjectures to systems of equations and we provide proofs in lower dimensions. To provide such counterparts we need to introduce a few novel concepts for system of equations. Part of this talk is based on joint works with Nassif Ghoussoub.

Stability Results for Measure Neutral Functional Differential Equations

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We consider a class of measure neutral functional differential equations whose integral form is given by

$$x(t) - x(0) = \int_0^t f(x_s, s)dg(s) + \int_{-r}^0 d_\theta[\mu(t,\theta)]x(t+\theta) - \int_{-r}^0 d_\theta[\mu(0,\theta)]\varphi(\theta),$$

and establish stability results using the correspondence between solutions of this equation and solutions of a generalized ordinary differential equation. We introduce the concept of regular stability of linear operators in a Banach space of $\mathbb{R}^n$-valued regulated functions. We discuss the total stability for a class of measure neutral functional differential equations.

Oscillation Criteria for Ordinary and Delay Half-Linear Differential Equations

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We study oscillatory properties of a half-linear differential equation of the form

$$(r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \quad \Phi(x) := |x|^{p-2}x, \ p > 1. \tag{1}$$

We introduce the so-called modified Riccati technique and present some oscillation results which can be obtained using this method. The method can be seen as a generalization of the transformation technique used in the linear case $p = 2$ and it enables us to compare oscillatory properties of two equations of type (1) with different coefficients and different powers in the nonlinearity. Namely, comparing equations with $p \neq 2$ and $p = 2$, we have that half-linear differential equations can be studied within the linear oscillation theory. We discuss also the case of the delay half-linear differential equation

$$(r(t)\Phi(x'))' + c(t)\Phi(x(\tau(t))) = 0, \quad \tau(t) \leqslant t.$$

New Results in the Theory of Discrete Search for Solutions of Equations and Inclusions in Metric Spaces

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We consider the existence and approximation problems for singularities of mappings between metric spaces, such as coincidences, common preimages of a closed subspace, common fixed points, and common roots. A new class of multivalued functionals strictly subordinated to convergent series is used for the solution of these problems. We give some examples and compare this new class of functionals with the so-called (α, β)-search functionals, $0 < \beta < \alpha$, introduced earlier by the author. Some new iteration schemes are suggested. Generalizations of several known results are obtained.

Let (X, ρ) be a metric space, $\mathbb{R}_+ = \{t \in \mathbb{R} \mid t \geqslant 0\}$ be the set of all nonnegative real numbers. Let also $P(A)$ stand for the total of all nonempty subsets of a set A, and the following convergent series be given:

$$\sum_{j=1}^{\infty} c_j < \infty,\ 0 < c_{n+1} < c_n,\ n \in \mathbb{N},\ S_k := \sum_{j=k}^{\infty} c_n, k \in \mathbb{N}. \tag{1}$$

Definition. A multivalued nonnegative functional $\varphi : X \to P(\mathbb{R}_+)$ *is strictly subjected to series* (1) on a metric space (X, ρ) if for any pair (x, t) of its graph $Gr(\varphi)$ there is a pair $(x', t') \in Gr(\varphi)$ such that $\rho(x, x') \leqslant t$, $(t > c_1) \Longrightarrow (\exists k \in \mathbb{N}, t' \leqslant c_k)$ and $(\exists k \in \mathbb{N}, t \leqslant c_k) \Longrightarrow (t' \leqslant \frac{t}{c_k} c_{k+1})$

The following statement demonstrates the local search principle for zeros of the considered functionals.

Theorem. *Let a multivalued functional $\varphi : X \to P(\mathbb{R}_+)$ be strictly subordinated to series* (1) *on a metric space X. Let also X be complete and the graph $Gr(\varphi)$ be* 0*-closed, i.e. contain all of its limit points of the form $(x, 0)$. Then for any pair $(x, t) \in Gr(\varphi)$ there is a sequence $\{(x_k, t_k)\}_{k=0}^{\infty} \subseteq Graph(\varphi)$ beginning from (x, t) and converging to some pair $(\xi, 0) \in Gr(\varphi)$. In addition, if there exists a pair $(x_0, t_0) \in Gr(\varphi)$ such that $t_0 \leqslant c_{k_0} \cdot \min\{1, \dfrac{r}{S_{k_0}}\}$ for some $k_0 \in \mathbb{N}, r > 0$, then there exists a corresponding limit pair $(\xi, 0) \in Gr(\varphi)$ with $\xi \in U(x_0, r)$.*

Several applications of this theorem to the existence and approximation problems of the above-mentioned singularities of mappings will be presented.

Structure of Dynamical Flow and Nonlocal Feedback Stabilization for Normal Parabolic Equations

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We study parabolic equations of so-called normal type for better understanding properties of equations of Navier–Stokes type.

By definition, a semilinear parabolic equation is called normal (NPE) if its nonlinear term defined by an operator B satisfies the following condition: $\forall v \in H^1$, the vector $B(v)$ is collinear to v. In other words, solutions to NPEs do not satisfy the energy estimate "in the most degree."

For the Burgers and the 3-D Helmholtz equations, we derive NPEs whose nonlinear terms $B(v)$ are orthogonal projections of nonlinear terms of the original equations on the straight line, generated by the vector v. The structure of the dynamical flow respondent to these NPEs will be described.

For the NPE corresponding to the Burgers equation, we construct nonlocal stabilization of solutions to zero by applying starting, impulse, or distributed feedback control supported in an arbitrary fixed subdomain of the spatial domain. The last result is applied to nonlocal stabilization of solutions for the Burgers equation.

Nonexistence of Monotone Positive Solutions to a Quasilinear Elliptic Inequality in a Half-Space

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Let $n \in \mathbb{N}$, $p, q \in \mathbb{R}$. We consider the partial differential inequality

$$-\Delta_p u \geqslant u^q \qquad (x \in \mathbb{R}^n_+). \tag{1}$$

Using a modification of the nonlinear capacity method for elliptic problems in half-spaces [1], we obtain the following result.

Theorem. *Let* $p > 1$ *and*

$$\max\{1, p-1\} < q < \frac{(n+1)(p-1)}{n-p+1}. \tag{2}$$

Then problem (1) *has no positive weak solutions* $u \in C^1(\mathbb{R}^n_+)$ *such that*

$$\frac{\partial u}{\partial x_n} \geqslant 0, \tag{3}$$

that is, $u(x', \cdot)$ *is monotone nondecreasing for each* $x' \in \mathbb{R}^{n-1}$.

Remark. Condition (3) is not accidental, since it is known that if $1 < p < 3$ (see [2] for $1 < p \leqslant 2$ and [3] for $2 < p < 3$) and $f : \mathbb{R}^1_+ \to \mathbb{R}^1_+$ is Lipschitz continuous, then condition (3) holds for all positive weak solutions of the following problem:

$$\begin{cases} -\Delta_p u = f(u) & (x \in \mathbb{R}^n_+), \\ u(x) = 0 & (x \in \partial\mathbb{R}^n_+) \end{cases} \tag{4}$$

such that $|Du| \in L^\infty(\mathbb{R}^n)$.

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Vector Differential-Difference Operators of Infinite Order in Spaces of Entire Functions of Exponential Type

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Let E be a complex Banach space, $\varphi(z) = \sum\limits_{n=0}^{\infty} C_n z^n$ be a formal power series whose coefficients are bounded linear operators in E, $h \in \mathbb{R}$, and $\sigma > 0$. We consider the infinite order differential-difference operator

$$(\varphi(S_h \frac{d}{dx})g)(x) = \sum_{n=0}^{\infty} C_n g^{(n)}(x + nh)$$

in the space $B_\sigma(E)$ of all entire E-valued functions of exponential type not greater than σ bounded on the real axis. Here, $S_h : B_\sigma(E) \to B_\sigma(E)$ is the translation operator,

$$(S_h g)(x) = g(x + h), \;\; g \in B_\sigma(E).$$

Such operators naturally arise while looking for entire solutions to the following simplest differential-difference equation in a Banach space:

$$u'(x) = Au(x - h) + f(x),$$

where A is a closed linear operator and f is an entire vector-valued function.

Theorem 1. *Suppose that $R > 0$ is the radius of convergence of the power series $\varphi(z)$, $g \in B_\sigma(E)$, and $u(x) = \sum\limits_{n=0}^{\infty} C_n g^{(n)}(x + nh)$. If $\sigma < R$, then this series converges uniformly on the real axis and $\varphi(S_h \frac{d}{dx})$ is a bounded linear operator in the space $B_\sigma(E)$.*

Let $A : D(A) \to E$ be a closed linear operator whose domain is not necessarily dense in E.

Theorem 2. *Let f be an entire function of exponential type not greater than σ bounded on the real axis. If $\operatorname{spec} A$ is the spectrum of the operator A, $0 \notin \operatorname{spec} A$ and $\sigma < min\{|\lambda| : \lambda \in \operatorname{spec} A\}$, then the equation $u'(x) = Au(x - h) + f(x)$ has a unique entire solution*

$$u(x) = -\sum_{n=0}^{\infty} A^{-(n+1)} f^{(n)}\left((x + (n+1)h)\right)$$

of exponential type at most σ bounded on the real axis, and this solution continuously depends on f in the topology of the space $B_\sigma(E)$.

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Some Estimates for Solutions to Higher Order Nonlinear Elliptic Equations in a Cylinder Domain

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The uniformly elliptic equation

$$\sum_{|\alpha|=|\beta|=m} D^\alpha(a_{\alpha\beta}(x)|D^m u|^{p-2} D^\beta u) = 0, \qquad p > 1, \tag{1}$$

with measurable and bounded coefficients is considered in a semi-cylinder

$$H = \{x \in \mathbb{R}^n : 0 < x_n < \infty, x' \in \Omega \subset \mathbb{R}^{n-1}\},$$

where $x = (x_1, \ldots, x_n) = (x', x_n)$, Ω is a bounded Lipschitz domain.

We study some properties of weak solutions at infinity. The main result is contained in the theorem below.

Theorem 1. *Let $u \in W^{m,p}_{loc}(H)$ be a weak solution of* (1) *subject to the homogeneous Neumann condition at the lateral side of the cylinder $\Gamma = \partial H \cap \{0 < x_n < \infty\}$. Let* *there exist weak derivatives* $D^\gamma u$ $(|\gamma| = 2m-1, \gamma_1 + \cdots + \gamma_{n-1} \leqslant m)$ *and* $\frac{\partial^l a_{\alpha\beta}(x)}{\partial {x_n}^l}$ $(l \leqslant m-1)$ *locally integrable in* H*. Let us also assume that*

$$\int\limits_{H\cap\{x_n<\rho\}} |D^m u|^p\,dx = o(\rho), \qquad \rho\to\infty.$$

Then there exist $\rho_0 > 0$ *and positive constants* C *and* η *such that the inequality*

$$\int\limits_{H\cap\{x_n>\rho\}} |D^m u|^p\,dx \leqslant Ce^{-\eta\rho}$$

holds $\forall\rho \geqslant \rho_0$.

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Solutions for a Semilinear Elliptic Equation in Dimension Two with Supercritical Growth

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We consider the problem

$$-\Delta u = \lambda u e^{u^p}, \quad u>0, \quad \text{in} \quad \Omega,$$
$$u = 0 \quad \text{on} \quad \partial\Omega,$$

where $\Omega\subset\mathbb{R}^2$ and $p>2$. Let λ_1 be the first eigenvalue of the Laplacian. For each $\lambda\in(0,\lambda_1)$, we prove the existence of solutions for p sufficiently close to 2. In the case where Ω is a ball, we also describe numerically the bifurcation diagram (λ, u) for $p>2$.

This work is in collaboration with Manuel del Pino and Monica Musso.

Maximal Regularity for Parabolic Problems with Dynamic Boundary Conditions

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We consider a mixed linear parabolic problem in the form

$$\begin{cases} D_t u(t,\xi) - A(\xi, D_\xi)u(t,\xi) = f(t,\xi), & t \in (0,T), \xi \in \Omega, \\ D_t u(t,\xi') + B(\xi', D_\xi)u(t,\xi') = h(t,\xi'), & t \in (0,T), \xi' \in \partial\Omega, \\ u(0,\xi) = u_0(\xi), \quad \xi \in \Omega, \end{cases} \tag{1}$$

where $A(\xi, D_\xi)$ is a second-order strongly elliptic operator, $B(\xi', D_\xi)$ is a suitable first-order operator. Here Ω is an open bounded subset of $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$. Let $p \in (1,\infty) \setminus \{\frac{3}{2}\}$. We determine necessary and sufficient conditions on f, h, and u_0 to ensure that problem (1) has a solution

$$u \in W^{1,p}((0,T); L^p(\Omega)) \cap L^p((0,T); W^{2,p}(\Omega))$$

such that $u|_{(0,T)\times\partial\Omega} \in W^{1,p}((0,T); W^{1-1/p,p}(\partial\Omega))$. We discuss the uniqueness of the solution, which is the case only for $p > \frac{3}{2}$.

Discrete Reaction-Diffusion Equations with Hysteresis

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In biological and population models, diffusive and nondiffusive substances often interact with each other according to hysteresis law. The simplest prototype model is a reaction-diffusion equation of the form

$$u_t = \Delta u + \mathcal{H}(u)$$

with $u = u(x,t)$, where $\mathcal{H}(u(x,\cdot))(t)$ $(= h_1$ or $-h_2)$ is the hysteresis operator (non-ideal relay) defined for each fixed $x \in \mathbb{R}^N$. Such a model has been studied since 1980s, mostly in the setting where the non-ideal relay $\mathcal{H}(u)$ is replaced by its multi-valued analog, which allows one to prove existence results but says little about the qualitative behavior of solutions even in the one-dimensional case.

In the talk, we will consider the discrete counterpart

$$\dot{u}_n = \frac{u_{n-1} - 2u_n + u_{n+1}}{\varepsilon^2} + \mathcal{H}(u_n), \quad n \in \mathbb{Z},$$

with $u_n = u_n(t)$ and $\varepsilon > 0$. This equation becomes well posed, and one can concentrate on the qualitative and quantative behavior of solutions. As the next step, this should allow one to properly pass to the continuous limit $\varepsilon \to 0$ and determine $u(x,t)$.

[1]jointly with Sergey Tikhomirov

Under appropriate initial conditions, we will see that the switching nodes $\mathcal{H}(u_n)$ form a certain pattern depending on h_1 and h_2, but surprisingly not on ε. In the exemplary case $h_2 = 0$, we will present the main ingredients that allow us to prove pattern formation.

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On the Chain Rule for the Divergence Operator in $\mathbb{R}^2$

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Given a bounded vector field $b\colon \mathbb{R}^d \to \mathbb{R}^d$, a scalar field $u\colon \mathbb{R}^d \to \mathbb{R}$, and a smooth function $\beta\colon \mathbb{R} \to \mathbb{R}$, we study the distribution $\mathrm{div}(\beta(u)b)$ in terms of $\mathrm{div} b$ and $\mathrm{div}(ub)$. In the case of BV vector fields b (and under some further assumptions), such characterization was obtained by L. Ambrosio, C. De Lellis and J. Malý, up to the residual term which is a measure concentrated on the so-called *tangential set* of b. We answer some questions posed in their paper concerning the properties of this term. In particular, we construct a nearly incompressible BV vector field b and a bounded function u for which this term is nonzero.

For stationary nearly incompressible vector fields b (and under some further assumptions) in the case where $d = 2$,we provide complete characterization of $\mathrm{div}(\beta(u)b)$ in terms of $\mathrm{div} b$ and $\mathrm{div}(ub)$. Our approach is based on the structure of level sets of Lipschitz functions on $\mathbb{R}^2$ obtained by G. Alberti, S. Bianchini and G. Crippa.

Extending our technique, we obtain new sufficient conditions for any bounded weak solution u of $\partial_t u + b\cdot\nabla u = 0$ to be renormalized, i.e. to solve also $\partial_t \beta(u) + b\cdot\nabla\beta(u) = 0$ for any smooth function $\beta\colon \mathbb{R} \to \mathbb{R}$. As a consequence, we obtain a new uniqueness result for this equation.

An Improved Level Set Method for Hamilton–Jacobi Equations

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In the classical level set method, a motion of an interface $\{\Gamma(t)\}_t$ in $\mathbb{R}^n$ is studied by representing the interface $\Gamma(t)$ as the zero level set of an auxiliary function $u(x,t)$, that is, $\Gamma(t) = \{x \in \mathbb{R}^n \mid u(x,t) = 0\}$, and solving the associated initial value problem of a partial differential equation for u. In this talk we are concerned with the case where the associated problem is given as the Hamilton-Jacobi equation of the form

$$\partial_t u(x,t) + H(x, \nabla u(x,t)) = 0 \quad \text{in } \mathbb{R}^n \times (0,T). \tag{1}$$

Here H is a geometric Hamiltonian and $\nabla u = (\partial_{x_i} u)_{i=1}^n$. The equation (1) is often called the *level set equation*. In practice, it might be difficult to compute the zero level set of u because the spatial gradient of u can be close to zero near $\Gamma(t)$ as time develops even if the initial gradient is large. To overcome this issue, we propose an *improved level set equation* of the form

$$\partial_t u(x,t) + H(x, \nabla u(x,t)) = u(x,t)G(x, \nabla u(x,t)). \tag{2}$$

Our goal is to prove that a solution u of (2) with a suitably defined G gives the same zero level set as (1), and that, globally in time, the slope of u is preserved near the zero level set.

We employ the theory of viscosity solutions to solve (2) since it is a nonlinear equation. To establish the preservation of the slope, we apply a comparison principle to the viscosity solution u of (2) and the signed distance function d to the interface, which is defined as

$$d(x,t) = \begin{cases} \operatorname{dist}(x, \Gamma(t)) & \text{if } u(x,t) \geqslant 0, \\ -\operatorname{dist}(x, \Gamma(t)) & \text{if } u(x,t) < 0. \end{cases}$$

Here $\operatorname{dist}(x, \Gamma(t)) = \inf\{|x-y| \mid y \in \Gamma(t)\}$. The distance function is known to be a solution of the eikonal equation $|\nabla d(x,t)| = 1$ in suitable senses. It is thus reasonable to use the signed distance function to guarantee that the slope of u remains one.

In order to define G in (2), we consider an evolution equation for d. If d is smooth near the interface, then it turns out that d satisfies (1) with $H(x - d(x,t)\nabla d(x,t), \nabla d(x,t))$ instead of $H(x, \nabla d(x,t))$ on the left-hand side. Thus, applying the Taylor expansion to H with respect to its first variable, we are led to the equation of form (2) with an error term. The function G is defined on the basis of this expansion.

Our main result is the following. Let u be the viscosity solution of (2) with the initial datum which equals $d(x,0)$ near $\Gamma(0)$. Assume that d is smooth near the interface. Then, for every $\varepsilon > 0$, there exists a constant $\rho = \rho(\varepsilon) > 0$ such that

$$\begin{cases} e^{-\varepsilon t} d(x,t) \leqslant u(x,t) \leqslant e^{\varepsilon t} d(x,t) & \text{if } 0 \leqslant d(x,t) \leqslant \rho(\varepsilon), \\ e^{\varepsilon t} d(x,t) \leqslant u(x,t) \leqslant e^{-\varepsilon t} d(x,t) & \text{if } -\rho(\varepsilon) \leqslant d(x,t) \leqslant 0. \end{cases}$$

This work is included in the author's Ph.D. thesis [1, Chapter 4].

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Dynamical Spike Solutions in a Nonlocal Model of Pattern Formation

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We analyze an integro-differential initial value problem obtained as a limit when the diffusion coefficient tends to infinity, from a system of reaction-diffusion-ODE equations

$$\begin{aligned} u_t &= f(u, \xi), && \text{for } x \in \Omega, t > 0, \\ \xi_t &= \int_\Omega g(u(x,t), \xi(t))dx, && \text{for } t > 0, \end{aligned}$$

with initial conditions $u(\cdot,0) = u_0 \in L^\infty(\Omega)$ and $\xi(0) = \xi_0 \in \mathbb{R}$. Such systems of equations arise, for example, from modeling of interactions between diffusing signaling factors and processes localized in cells and on cell membranes. For the reduced models, we show that nonlocal terms induce a destabilization of stationary solutions, which may lead to a blow-up of spatially inhomogeneous solutions, either in finite or infinite time.

The presentation is based on joint work with Grzegorz Karch (University of Wroclaw), Anna Marciniak-Czochra (University of Heidelberg), and Kanako Suzuki (Ibaraki University).

Numerical simulation of energy balance models in global climate applications

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In this work, we develop and apply high-order finite volume schemes to numerical simulation of energy balance models in the context of global climate. We consider a coupled model atmosphere-ocean taking into account mechanisms of exchange of energy. We focus our attention on two particular applications. First we consider the effect of the deep ocean on the surface, evidencing the thermostatic effect of the ocean. The second application concerns land-sea distribution. In this case, the results are compared with those obtained with only continental part and only oceanic part. The numerical approach uses Weighted Essentially Non Oscillatory reconstruction in space and third order Runge–Kutta TVD scheme for time integration.

This work has been partially supported by the Spanish company REPSOL, under agreement with Universidad Politécnica de Madrid. The author is also greateful to the Spanish Ministry of Economy and Competitiveness for its financial support through the project SCARCE (Consolider-Ingenio 2010 CSD2009-00065).

Pohozaev's Identity and Compact Supported Solutions

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We discuss a method of studying compact supported solutions for elliptic equations with singular and non-Lipschitz nonlinearities. The method is based on variational methods where a crucial role is played by the Pohozaev identity. The main novelty is employment of the Pohozaev identity [4] to prove of the existence of compact supported solutions. Furthermore, a new type of critical exponents of the considered problems are obtained by application of the spectral analysis [1, 3] with respect to Pohozaev's fibering method [5]. The talk is based on joined works with Yu. V. Egorov, P. Takac, J. I. Diaz, J. Hernandez [2, 6, 7].

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Minimum-Thrust Problem as the Beginning of Interplanetary Low-Thrust Trajectories Design

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One of main problems in optimization of trajectories for spacecraft with finite-thrust is the absence of the existence theorem. This is why it is so difficult to design a robust and efficient numerical optimization technique. Indeed, if a numerical scheme does not converge to a solution, then the real reason is unknown since it can be either the absence of a solution or the numerical scheme failure. Therefore, identification of the boundary of the solution region is an actual problem. In this way, solving the minimum-thrust problem gives necessary assessement.

The minimum-thrust problem is formulated similar to any other problem with low-thrust switchings, but the functional to minimize is now thrust. It is supposed that the thrust can be either maximal or zero, the maximal thrust magnitude is invariable along trajectory, and its direction is unconstrained. The time transfer is fixed.

This problem is dual to the minimum-time problem, the necessary optimality conditions for both problems being the same, and the transversality condition used for the thrust magnitude is linked to the transversality condition for time, which is the normality condition. That is, the solution of the minimum-thrust problem corresponds to the minimum-time trajectory for the minimal thrust magnitude providing the transfer duration.

We solve the optimization problems using Pontryagin's maximum principle. The boundary-value problem is solved using the continuation (homotopic) method.

As an example, the problems of interplanetary trajectories for spacecraft with minimum-thrust and with thrust switchings optimization are solved.

On Attractors of m-Hessian Evolutions

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Let $u \in C^{4,2}(\bar{Q})$, $\Omega \subset R^n$, $Q = \Omega \times (0; \infty)$, and u_{xx} be the Hesse matrix of u in space variables. We denote by $T_p[u] = T_p(u_{xx})$, $1 \leqslant p \leqslant n$, the p-trace of u_{xx} and introduce a p-Hessian evolution operator by setting $E_p[u] := -u_t T_{p-1}[u] + T_p[u]$. We say that an evolution u is m-admissible in $\bar{Q}_T$ if $u \in C_m(\bar{Q}_T)$, where

$$C_m(\bar{Q}_T) := \{u \in C^{2,1}(\bar{Q}), E_p[u] > 0, p = 1, \dots, m\}. \tag{1}$$

We investigate asymptotic behavior of solutions to following initial-boundary value problems

$$E_m[u] = f, \quad u|_{\partial' Q_T} = \phi, \quad 1 \leqslant m \leqslant n, \tag{2}$$

where $\partial' Q_T = \Omega \cup \partial\Omega \times [0; T]$. In particular, we have obtained the following statement.

Theorem 1. *Let $f > 0$, $f, \phi \in C^{2,1}$, $\phi = 0$ on $\partial\Omega \times [0; \infty)$, and $\partial\Omega \in C^2$. Assume that $\lim_{t\to\infty} f(x,t) = \bar{f}(x)$ and there exists a C^2-solution $\bar{u}$ to the problem $T_m[u] = \bar{f}$. Then any solution $u = u(x,t)$ to problem (2) converges to the function $\bar{u}(x)$ uniformly in C. Moreover, this solution is an m-admissible evolution, i.e. $u \in C_m(\bar{Q}_T)$ for all $T > 0$.*

To emphasize the second assertion of Theorem 1, we formulate the following non-existence theorem.

Theorem 2. *Suppose that there is a point $x_0 \in \Omega$ such that $\phi(x, 0)$ does not satisfy (1) at x_0. Then problem (2) has no solution in $C^{2,1}$ whatever the function $f > 0$ is.*

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Systems of Functional-Differential Equations Modelling the Dynamics of Structured Populations — Modelling, Analysis and Applications

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Mathematical model systems for the dynamics of structured populations are deserving more attention since information on important factors of their dynamics is substantially growing, e.g., in life sciences, social sciences, and economics. Advanced experimental and information technology are producing an enormous amount of data, e.g., on a large spectrum of processes in individual cells. The detailed information on single agents has to be transferred to populations, which by itself is a challenge to mathematical analysis. This transition may lead to dynamical systems for densities with respect to structural variables. There is an urgent demand for the analysis of and numerical methods for the resulting systems of functional-differential equations with multiple structural variables.

In this lecture, we are starting from an investigation of drug resistance in malaria where the parasites consist of two types distinguished by different reactions to a specific drug: drug sensitive and drug resistant parasites. Their dynamics is not explicitly modelled. Their influence is represented via corresponding structural variables in the infected populations.

Motivated by this, we develop a generalized mathematical model to describe this structured population dynamics. The analysis of the model, which is a system of finite integro-partial differential equations with its complex implicit boundary conditions, is presented. By expanding the method of characteristics, we prove the existence and uniqueness of a solution, as well as its positivity.

Finally, in an overview on arising challenges, the reduction of high dimensional systems to systems which are simpler to be calibrated and numerically solved, will be discussed.

The lecture is partially based on joint research with Le Thi Thanh An and Maria Neuss-Radu.

Power-Logarithmic Solutions for Nonlinear Elliptic Systems

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We consider the higher order nonlinear elliptic system

$$\mathcal{L}u \equiv \operatorname{div}^t A(x, D^s u) = 0$$

in $\mathbb{R}^n$, where $s+t$ is even, under structure conditions providing coercivity and monotonicity of $\mathcal{L}$ in pair with $\Delta^{(s-t)/2}$ in the space $H^s = W_2^s(\mathbb{R}^n)$. Our class of systems includes divergent systems with standard structure conditions as well as Cordes-type equations.

It is known that solutions possess correct behavior on a segment of the scale of weighted spaces H_a^s with the norm $\|D^s u; L_2(\mathbb{R}^n; |x|^a)\|$, $a \in (a_*, a^*) \subset\subset (-n, n)$, where a_*, a^* are defined by the modulus of ellipticity of the system. This means that the only possible singularities of solutions with order of singularity in the segment (a_*, a^*) coincide with the singularities of the fundamental solution to $\Delta^{(s+t)/2}$ or one of its derivatives.

We discuss the end-point situation $a = a_*$, $a = a^*$. We establish power and power-logarithmic estimates for solutions (depending on whether or not the corresponding moment of $\mathcal{L}u$ is zero) and give an example of equation with solutions of both power and power-logarithmic types.

Quasiergodic Hypothesis and Arnold Diffusion in Dimension 3 and 4

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The famous ergodic hypothesis claims that a typical Hamiltonian dynamics on a typical energy surface is ergodic. However, KAM theory disproves this. It establishes a persistent set of positive measure of invariant KAM tori. The (weaker) quasi-ergodic hypothesis proposed by Ehrenfest and Birkhoff says that a typical Hamiltonian dynamics on a typical energy surface has a dense orbit. This question remains open. In the early 1960s, Arnold constructed an example of instabilities for a nearly integrable Hamiltonian of dimension $n > 2$ and conjectured that this was a generic phenomenon nowadays called Arnold diffusion. In the last two decades, a variety of powerful techniques to attack this problem were developed. In particular, Mather discovered a large class of invariant sets and a delicate variational technique to shadow them. In a series of preprints, one jointly with P. Bernard and K. Zhang, and two jointly with K. Zhang, we prove a strong form of Arnold's conjecture in dimension $n = 3$ and 4. Jointly with M. Guardia, we also prove a weak form of quasiergodic hypothesis for $n = 3$.

Stability of Autoresonance under Persistent Perturbation by White Noise

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Mathematical models of capturing in resonance are reduced to some systems of ordinary differential equations by two scale method. In this way the problem of stability of such resonance phenomena is reduced to the stability of equilibrium of a nonlinear nonautonomous system. Specific to autoresonance models is the local Lyapunov stability with respect to a deterministic perturbation. It is well known that there is no Lyapunov type stability with respect to white noise in such systems. However, the autoresonance is stable in the physical sense. What does it means in mathematics? This question is discoursed for a general system of ordinary differential equations.

The main object is the deterministic system

$$\frac{d\mathbf{y}}{dT} = \mathbf{a}(\mathbf{y}, T), \quad \mathbf{y} \in \mathbb{R}^n, \ T > 0. \tag{1}$$

The point $\mathbf{y} = 0$ is an equilibrium, namely $\mathbf{a}(0, T) \equiv 0$. Let perturbed system be the stochastic Ito equation

$$d\mathbf{y} = \mathbf{a}(\mathbf{y}, T)dT + \mu B(\mathbf{y}, T)\, d\mathbf{w}(T), \ T > 0; \quad \mathbf{y}|_{T=0} = \mathbf{x}, \ 0 < \mu^2 \ll 1. \tag{2}$$

Here $\mathbf{w}(T)$ accounts for the standard Brownian motion in $\mathbb{R}^n$, $B(\mathbf{y}, T)$ is a matrix of size $n \times n$ with the property $B(0, T) \not\equiv 0$. The solution $\mathbf{y} = \mathbf{y}_\mu(T; \mathbf{x})$ is a stochastic process in $\mathbb{R}^n$ depending on both the initial point $\mathbf{x}$ and a small parameter μ.

Let the equilibrium position $\mathbf{y} = 0$ of system (1) be asymptotically stable in the sence of Lyapunov. Consider the problem of stability under white noise as follows: *does the trajectory* $\mathbf{y} = \mathbf{y}_\mu(T; \mathbf{x})$ *remain near equilibrium if the perturbations* μ *and* $|\mathbf{x}|$ *are small and the matrix belongs to a fixed ball* $||B|| < M$?

It is well known that if $B(0, T) \not\equiv 0$, then there is no Lyapunov type stability with respect to white noise. Almost all trajectories of the perturbed system diverge from the equilibrium to arbitrarily large distances in finite time. The concept of stability must be modified in appropriate way. The similar problem for autonomous systems was considered by M. Freidlin. We solve the problem for nonautonomous systems by the parabolic equation method [1].

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Nonlinear Stationary Phase and Asymptotic Stability of the Toda Lattice

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The asymptotic analysis of integrable systems (or soliton equations) can often be reduced to the asymptotic analysis of Riemann–Hilbert factorization problems in the complex plane or a complex variety.

This is achieved through the Riemann–Hilbert deformation method, which involves complex and harmonic analysis, Riemann surface theory and potential theory. It can be seen as a nonlinear (or non-commutative) analogue of the methods of stationary phase and steepest descent.

In this talk, I will review the basic ideas of the Nonlinear Stationary Phase method in a general hyperelliptic Riemann surface, using the Toda lattice as a model.

Spectral Flow of Families of Dirac Operators, Index Theorem, and Creation of Electron-Hole Pairs in Graphene

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We compute, in topological terms, the spectral flow of an arbitrary family of self-adjoint Dirac type operators with classical (local) boundary conditions on a compact Riemannian manifold with boundary under the assumption that the initial and terminal operators of the family are conjugate by a bundle automorphism. This result is used to study conditions for the existence of nonzero spectral flow of a family of self-adjoint Dirac type operators with local boundary conditions in a two-dimensional domain with nontrivial topology. Possible physical realizations of nonzero spectral flow are discussed.

Periodic Maxwell's Operators with Preassigned Gaps in Spectrum[1]

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We study the Maxwell operator in two dimensional dielectric medium with small heterogeneous inclusions that are periodically distributed with a small period ε. Media with such a structure are typical for photonic meta-materials being artificial composite materials with required electromagnetic properties. Spectrum of the Maxwell operator in such a medium is continuous and can have gaps, but their presence is not guarantied. On the other hand, it is an important application in radio-engineering to know the location of gaps in the spectrum. That is why our main purpose is to construct inclusions which provide existence of preassigned gaps in the spectrum.

We consider the traps-like inclusions that are the annuli of the completely conducting material with slim slits. We prove that for sufficiently small ε the spectrum of the Maxwell operator is a finite gap with the edges converging to given numbers as $\varepsilon \to 0$. We establish a one-to-one correspondence between parameters of the traps-like inclusions and the edges of the limiting spectrum.

[1]Joint works with A. Khrabustovskyi.

Instabilities in Rotational MHD Flows: A Comprehensive Short-Wavelength Analysis

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We perform a local stability analysis of rotational flows in the presence of a constant vertical magnetic field and an azimuthal magnetic field with a general radial dependence characterized by an appropriate magnetic Rossby number.

Employing the short-wavelength approximation, we develop a unified framework for investigation of the standard, the helical, and the azimuthal version of the magnetorotational instability, as well as of current-driven kink-type instabilities.

Considering the viscous and the resistive setup, we mainly focus on the case of small magnetic Prandtl numbers applied, e.g., to liquid metal experiments but also to the colder parts of accretion disks. We show in particular that the inductionless versions of MRI that were previously thought to be restricted to comparably steep rotation profiles extend well to the Keplerian case if only the azimuthal field slightly deviates from its field-free profile.

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Asymptotic Properties of Arnold Tongues and Josephson Effect

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Consider the equation

$$\frac{dx}{dt} = \frac{\cos x + a + b\cos t}{\mu}, \qquad (t, x) \in \mathbb{R}^2/2\pi\mathbb{Z}^2 = \mathbb{T}^2, \tag{1}$$

arising as a model for the Josephson junction in external oscillating electromagnetic field. Here a and b depend on parameters of external field and current applied to the junction, and μ is a characteristic of the junction, so it is natural to let $\mu = \mathrm{const}$ and vary a and b. From mathematical point of view, the dependence between current and voltage is expressed in terms of the rotation number of equation (1).

Recall that *Arnold tongues* are level sets $E_{\rho_0} = \{(a, b) : \rho(a, b) = \rho_0\}$ of the rotation number on the plane of parameters (a, b) that have nonempty interior. For a generic family of vector fields Arnold tongues exist for all $\rho_0 \in \mathbb{Q}$.

In family (1), Arnold tongues exist only for $\rho_0 \in \mathbb{Z}$. The reason is that the flow map $P_{a,b}$ in the coordinate $u = \tan(x/2)$ is Möbius over the period, whereas a Möbius map has periodic points of period greater than 1 iff it is periodic itself.

Moreover, there is the symmetry $(x, t) \to (-x, -t)$ of equation (1), hence $P_{a,b}$ has the symmetry $P_{a,b}(x) = -P^{-1}_{a,b}(-x)$. This means that $P_{a,b}$ has two fixed points $\pm z$ inside the tongues, and on the boundary of the tongue these points coincide either at 0 or at π. Let $a_{0,k}(b)$ and $a_{\pi,k}(b)$ denote the corresponding values of a on the boundary of E_k. It can be shown that monotonicity in a implies that $a_{\dots,k}(b)$ are well-defined for every $b \geqslant 0$.

We study asymptotics of the functions $a_{\dots,k}(b)$ as $b \to +\infty$. It appears that they are asymptotically equivalent to the Bessel functions (after appropriate shift and scaling).

Theorem 1. *There exist positive constants $C'_1, C'_2, K'_1, K'_2, K'_3$ such that the estimates*

$$\left|\frac{a_{0,k}(b)}{\mu} - k + \frac{1}{\mu} J_k\left(-\frac{b}{\mu}\right)\right| \leqslant \frac{1}{b}\left(K'_1 + \frac{K'_2}{\mu^3} + K'_3 \ln\left(\frac{b}{\mu}\right)\right),$$
$$\left|\frac{a_{\pi,k}(b)}{\mu} - k - \frac{1}{\mu} J_k\left(-\frac{b}{\mu}\right)\right| \leqslant \frac{1}{b}\left(K'_1 + \frac{K'_2}{\mu^3} + K'_3 \ln\left(\frac{b}{\mu}\right)\right)$$

hold as soon as the parameters b, μ and a number $k \in \mathbb{Z}$ satisfy the inequalities

$$|k\mu| + 1 \leqslant C'_1 \sqrt{b\mu}, \qquad b \geqslant C'_2 \mu.$$

Here $J_k(t)$ is the Bessel function of the first kind.

The talk is based on the joint work [1] with Olga Romaskevich.

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On Global Attractors of Hamilton Nonlinear PDEs

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Our main goal is a survey of our results [1–13] and problems in the theory of global attractors of Hamilton nonlinear PDEs in infinite space. Main results mean the following *global attraction*:

[1]Partially supported by Alexander von Humboldt Research Award, and the grant of Russian Foundation for Basic Research.

each finite energy solution converges to a finite-dimensional attractor A in the sense of a local convergence as $t \to \pm\infty$.

The structure of the global attractor crucially depends on the symmetry group of the equation.

A. For a generic equation, the attractor A is the set of all stationary states $\psi(x)$.

B. For generic $U(1)$-invariant equations, the attractor A is the set of all "solitary waves", $e^{-i\omega t}\psi(x)$.

C. For generic translation-invariant equations, the attractor A is the set of all solitons $\psi(x-vt)$.

The global attraction in the cases **A**, **B**, and **C** give the first mathematical model of Bohr's transitions between quantum stationary states and wave-particle duality respectively.

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Solution of Main Hydromechanics Equations

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The main classical equations of hydromechanics are the Euler equations and the Navier–Stokes ones. These equations are of interest from purely mathematical point of view and have numerous applications to practical problems. For today, many of the issues associated with these equations are still not solved and need more detailed investigation. It applies especially to the $3-D$ versions of the equations. One of the main problems is the lack of a constructive method of solution. The development of such a method is an important and an interesting task.

The author proposes a solution to this problem for the case of $3-D$ incompressible fluid flow. The major unknowns for this case are velocities u, v, w and pressure p. The original system contains four equations. Since the Euler equations is a particular case of the Navier–Stokes ones as $Re \to \infty$, the main attention is paid to the Navier–Stokes equations. All of the changes in the fair way to the Navier–Stokes equations, except for the last one, are valid for the Euler equations.

The main idea of the proposed approach is the following. The solution of the original equations is reduced to the aggregate of much more simple tasks. The whole chain of transformations can be divided into two steps, the first integration and the second one.

The first integration is based on the transformation of each of the original equations to the divergence form

$$\frac{\partial P_i}{\partial x} + \frac{\partial Q_i}{\partial y} + \frac{\partial R_i}{\partial z} + \frac{\partial S_i}{\partial t} = 0, \quad (1)$$

where P_i, Q_i, R_i, S_i are some combinations of the major unknowns and their first derivatives with respect to coordinates.

Each equation of this type admits a solution. Combining these solutions for each of the original equations and partially excluding non-divergence terms, we can get a first integral of the Navier–Stokes equations. So we deal with nine relations between the major unknowns unknown u, v, w, p, the associated unknowns Ψ_j, where $j = 1, 2, \ldots, 15$, and arbitrary additive functions of three variables α_i, β_i, γ_i, δ_i.

The second integration begins with the analysis of relations representing the first integral. There are nine of them. Four of them are linear. They define the structure formulas for the major unknowns u, v, w, and p. The remaining five relations are nonlinear. They specify separate fragments in the expressions for u, v, w, and p. Considering these five nonlinear equations, we find the pattern. These equations can be obtained under some additional conditions only. We have two compatibility conditions in the form of fifth order equations with respect to the unknowns Ψ_j, where $j = 1, 2, ..., 9$. Each series of nine functions Ψ_j satisfying these equations generates a solution of the Navier–Stokes equations. The values of all relevant quantities are strictly defined. So the values of the unknowns Ψ_j, where $j = 13, 14, 15$, can be set arbitrarily. The unknown Ψ_j with $j = 10, 11, 12$ are determined from the system of three inhomogeneous equations that can be resolved without any restrictions. As a result, all the unknowns are found and the problem is completely resolved.

Solution of the Euler equations at this stage have some differences. If $\frac{1}{Re} \to 0$, then the considered equations become fourth-order ones. Solution of these equations with respect to the unknowns Ψ_j requires a special approach.

Thus, we are able to construct solutions to the Navier–Stokes equations and the Euler equations while fully preserving the nonlinear terms. The solution obtained in this way contain arbitrary functions of three and four variables.

The approach described above can be implemented in general case, without any limitations of private manner.

Asymptotic Stability of Solitons for Nonlinear Hyperbolic Equations

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The first results on asymptotic stability of solitary waves were established by Soffer and Weinstein [9, 10] for small solutions to the nonlinear Schrödinger equations with a potential and $U(1)$-invariant nonlinearity whose linear part vanishes at the origin. This strategy was further developed by Pego and Weinstein in 1992–1997, by Tsai and Yau in 2002–2007, by Martel, Merle, Mizumachi, and Tsai in 1990–2011, and others.

The asymptotic stability of the solitons was proved first by Buslaev, Perelman, and Sulem [2, 3] for the 1-D nonlinear translation invariant Schrödinger equations.

The presence of the zero eigenvalue in the spectrum of the linearized dynamics prohibits the application of the standard Lyapunov theory. The proofs rely on a novel general strategy. The principal ingredients of the strategy are the following: modulation equations for symplectic projection of the trajectory onto the solitary manifold, dispersion decay in the symplectic orthogonal directions to the solitary manifold, the method of majorants, and Poincaré normal forms. The novel condition referred to as the Fermi Golden Rule provides the strong coupling of the discrete eigenmodes with the continuous spectrum.

In all of the above mentioned papers, the spectral conditions and the Fermi Golden Rule were postulated but the examples were not constructed. In [1, 6], we considered the Schrödinger equation coupled with the $U(1)$-invariant nonlinear oscillator which provide all spectral conditions needed .

This general strategy was developed in [4, 5] for the proof of asymptotic stability of solitons for a classical particle coupled with the Schrödinger and the Dirac equations.

The first results for relativistic nonlinear equations were established in [7, 8]. The asymptotic stability has been proved for the kinks of the 1-D nonlinear wave equations with Ginzburg–Landau potentials.

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The Liouville Theorem for the Steady Navier–Stokes Problem in Axially Symmetric 3D Spatial Case

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We study the Navier–Stokes equations of steady motion of a viscous incompressible fluid in $\mathbb{R}^3$. We prove that these equations do not have nontrivial solutions in the whole space $\mathbb{R}^3$ in the axially symmetric case with no swirl. We also present for this case a short proof of the existence theorem for the boundary value problem for the steady Navier–Stokes equations in a three-dimensional exterior domain with multiply connected boundary.

Optimization Methods for Nonlinear Model Predictive Control of Non-Stationary Partial Differential Equations

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Many spatio-temporal processes in natural sciences, life sciences, and engineering are described by non-stationary partial differential equations (PDE). It would be of high practical relevance as well as a mathematical challenge to use such models for a process optimization subject to numerous important inequality restrictions. However, in the presence of disturbances and modeling errors, the real process will never follow the off-line computed optimal solution. Thus, the challenge is to compute feedback control taking these perturbations into account. Probably the most powerful realization is Nonlinear Model Predictive Control (NMPC), which is based on the two following steps: a simultaneous on-line estimation of the system state and parameters and re-optimization of the optimal control for the current parameter and state values. The challenge is to solve these optimal control problems with high frequency in real time. Whereas for ordinary differential equations significant progress has been made during the last decade, the NMPC problem for PDE is far from being solved.

We present a new optimization method for NMPC for PDE models based on innovative multi-level iterations strategy to make NMPC computations real-time feasible for PDE optimal control problems. The talk is based on joint work with H. G. Bock, G. Kriwet, and J. P. Schloeder.

On Multi-Weighted Parabolic Initial Boundary-Value Problems

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In a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$, an elliptic boundary value problem

$$(A + \lambda I)\, u = f \quad \text{in } \Omega, \quad Bu = 0 \text{ on } \partial\Omega \tag{1}$$

is considered. Here A is an elliptic partial differential operator of even order, B is an operator of boundary conditions and λ is the spectral parameter. In 1962, S. Agmon introduced a condition on the principal symbols of A and B, called the condition of having "the ray of minimal growth of the resolvent." More precisely, under this condition, the resolvent operator $R(\lambda) : f \to u$ solving problem (1) is a bounded operator in the space $L_2(\Omega)$ and

$$\|R(\lambda)\| \leqslant const\, (|\lambda| + 1)^{-1} \tag{2}$$

for large enough modulo λ belonging to a certain ray on the complex plane, emerging from the origin. R. Seeley called this condition "Agmon's condition" and proved (2) for elliptic systems of even order. Agranovich and Vishik (1964) investigated operators polynomially dependent on λ under the condition called "ellipticity with a parameter." The estimate of type (2) was obtained by Grubb in 1996 under the corresponding parameter-ellipticity condition for pseudodifferential boundary value problems.

The parameter-ellipticity condition for matrix Douglis–Nirenberg elliptic operators acting on compact manifolds without boundary was introduced by the author in 1972, and estimate (2) was proved for $R(\lambda)$ acting in an appropriate Sobolev space. The parameter-ellipticity condition was elaborated further by the author as well as by R. Denk, M. Feierman, R. Mennicken, and L. Volevich in a series of papers (1995–2013) but estimate (2) was obtained only for very special cases.

This work is devoted to problem (1) with a Douglis–Nirenberg operator A. The aim of the work is to prove estimate (2) for $R(\lambda)$ acting in an appropriate Sobolev space. Using (2), we prove a unique solvability of the corresponding t-dependent initial-boundary value problem

$$(\partial/\partial t + A)\, u = f \quad \text{in } \Omega, \quad u|_{t=0} = 0, \quad Bu = 0 \text{ on } \partial\Omega$$

in appropriate anisotropic Sobolev–Slobodetskii spaces. The problem is not parabolic in the usual sense but it may be naturally called "multi-weighted parabolic" according to author's paper of 2013, where the boundaryless case was investigated.

Hadamard Type Asymptotics for Eigenvalues of the Neumann Problem for Elliptic Operators

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This talk considers how the eigenvalues of the Neumann problem for an elliptic operator depend on the domain. The proximity of two domains is measured in terms of the norm of the difference between the two resolvents corresponding to the reference domain and the perturbed domain, and the size of eigenfunctions outside the intersection of the two domains. This construction enables the possibility of comparing both nonsmooth domains and domains with different topology.

An abstract framework is presented. The main result is an asymptotic formula where the remainder is expressed in terms of the measure of proximity described above provided the last one is sufficiently small.

As an application, we consider the Laplacian in both $C^{1,\alpha}$ and Lipschitz domains. For the $C^{1,\alpha}$ case, an asymptotic result for the eigenvalues is given together with estimates for the remainder. We also provide an example demonstrating the sharpness of the obtained result. For the Lipschitz case, the proximity of eigenvalues is estimated.

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Conservation Laws of Generalized Billiards that are Polynomial in Momenta

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We deal with dynamics of particles moving along a Euclidean n-dimensional torus or an n-dimensional parallelepiped under the force whose potential is proportional to the characteristic function of a region D with regular boundary. After reaching this region, the trajectory of the particle is refracted according to the law resembling the Snell–Descartes law of geometrical optics. When the energies are small, the particle does not reach the region D and elastically bounces off its boundary. In this case, we obtain a dynamical system of billiard type (which was intensely studied with respect to strictly convex regions). In addition, we discuss the problem of the existence of nontrivial first integrals that are polynomials in momenta with summable coefficients and are functionally independent of the energy integral. Conditions on the boundary of the region D under which the problem does not admit nontrivial polynomial first integrals are found. Examples of nonconvex regions are given. For these regions, the corresponding dynamical system is obviously nonergodic for fixed energy values (including small ones), however, it does not admit polynomial conservation laws independent of the energy functional.

Model Validation and Optimum Experimental Design of Partial Differential Equations

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Mathematical models are of great importance for manufacturing and engineering. Besides providing a scientific insight into processes, the mathematical models are used in process optimization and control. However, the results from simulation and optimization will only be reliable if an underlying model precisely describes a given process. This implies a model validated by experimental data with sufficiently good estimates for model parameters.

Often many expensive experiments have to be performed for estimating the parameters in order to get enough information for parameter estimation. The number of experiments can be drastically reduced by computing optimal experiments.

Our method is based on the direct multiple shooting time domain decomposition. The states of the partial differential equation are infinite dimensional, and thus the discretization leads to high dimensional multiple shooting node values. In the parameter estimation problem, we reduce the degree of freedom by an efficient condensing technique. For the error assessment we need the confidence region of the parameters, which can be computed from the reduced parameter estimation problem.

The optimum experimental design problem is based on the reduced parameter estimation problem. The high dimension of the design space makes a forward condensation useless. Instead we use an adjoint approach and a computable approximation of the Hessian.

Small Oscillations in Space-Multidimensional Hamiltonian PDE

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I will start my talk with discussing predictions concerning long-time behavior of small solutions for non-linear hamiltonian PDEs on $\mathbb{T}^d$, given by the KAM and the weak turbulence theories, and will explain difficulties in rigorous verification of these predictions. Next I will present a sufficiently complete KAM-theory of small oscillations in nonlinear beam equation in $\mathbb{T}^d$, $d \geqslant 1$,

$$u_{tt} + \Delta^2 u + mu + g(x, u) = 0\ , \quad t \in \mathbb{R},\ x \in \mathbb{T}^d, \tag{1}$$

where $g(x, u) = 4u^3 + O(u^4)$. This theory states that, for generic m, most of the small amplitude invariant finite dimensional tori of the linear equation (1) $|_{g=0}$ persist as invariant tori of the nonlinear equation (1). If $d \geqslant 2$, then not all of the persisted invariant tori are linearly stable, and we present explicit examples of those which are partially hyperbolic. The set $\mathfrak{A}$ of the constructed finite-dimensional invariant tori for Eq. (1) is "asymptotically dense at the origin O of the function space" in the sense that $\mathfrak{A}$ intersects outside O any open cone in the function space which has the vertex at O.

This is a joint work with Håkan Eliasson and Benoît Grebert.

Upper Hausdorff Dimension Estimates for Invariant Sets of Evolutionary Variational Inequalities

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We investigate a class of evolutionary variational inequalities in a general rigged Hilbert space structure for which the solution is not unique for a given initial state. Under some conditions, these variational inequalities generate multivalued semi-flows which can have semi-invariant compact sets. Using techniques similar to those of O. A. Ladyzhenskaya, A. Douady, I. Oesterlé and R. Temam, we derive upper bounds for the fractal dimension of such sets. We show that it is possible for noninjective maps to use the information about the degree of noninjectivity in order to get dimension estimates under weaker conditions than before. As an example, we consider a PDE problem from continuum mechanics with dry friction.

Coarsening Dynamics of Droplets Driven by Thin-Film Type Equations

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The goal of this study is the reduction of the dynamics governed by thin film type equations onto an “approximate” finite-dimensional invariant manifold. The reduction corresponds to the physical situation of the late phase evolution of thin liquid films dewetting on a solid substrate, where arrays of drops connected by an ultrathin film of thickness ε undergo a slow-time coarsening dynamics. With this situation in mind, we construct an asymptotic approximation of the corresponding invariant manifold parametrized by a family of droplet pressures and positions in the limit as $\varepsilon \to 0$.

Moreover, reduced systems of ODEs governing the dynamics on the manifold are derived for different slip regimes considered at the solid substrate. Subsequently, dependence of the coarsening rates (i.e. the law describing how fast the number of drops decreases in time) on the physical parameters such as inertia, viscosity of the film as well as slippage is analyzed. In the limiting case of free suspended films, the existence of a threshold is shown for the decay of initial distributions of droplet distances at infinity at which the coarsening rates switch from algebraic to exponential ones.

A Lemma on Partial Shadowing and Its Applications to Non-Hyperbolic Dynamics[1]

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Shadowing is one of classical problems in the modern theory of dynamical systems. In this talk, we are going to demonstrate that a very weak version of shadowing holds true for general homeomorphisms of compact metric spaces.

Definition 1. Let X be a metric space with metrics d, $T : X \to X$ be a homeomorphism, and $\delta > 0$. A sequence $p_k \in X$ ($k \in \mathbb{Z}$) is called δ-pseudotrajectory if we have $d(T(p_k), p_{k+1}) \leqslant \delta$ for any $k \in \mathbb{Z}$.

Definition 2. We say that a homeomorphism T satisfies the shadowing property if for any $\varepsilon > 0$ there exists $\delta > 0$ such that given a δ-pseudotrajectory p_k, there is a point $x \in X$ such that $d(T^k(x), p_k) \leqslant \varepsilon$, $k \in \mathbb{Z}$.

Of course, one cannot expect that a general homeomorphism (or even diffeomorphism) of the space X satisfies shadowing property. The majority of shadowing criteria link this property with hyperbolicity and structural stability of the mapping T. Nevertheless, the following is true.

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Theorem 1. *Let X be a compact metric space, $T : X \to X$ be a continuous mapping. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any δ-pseudotrajectory p_k $(k \geqslant 0)$ of the mapping T there exist a sequence $k_j \to \infty$ and a point $x_0 \in M$ such that $d(p_{k_j}, T^{k_j}(x_0)) \leqslant \varepsilon$ as $j \in \mathbb{N}$.*

In a nutshell, given a pseudotrajectory, we can select a subsequence shadowed by a subsequence of the precise trajectory with the same indices.

Krylov-Based Methods for Approximate Solving of Differential Equations and Functional Calculus

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We consider the problem

$$Fx'(t) = Gx(t) + bu(t), \qquad y = \langle x(t), d\rangle. \tag{1}$$

Here F and G are complex $n \times n$-matrices, $b, d \in \mathbb{C}^n$, $\langle \cdot, \cdot\rangle$ is the inner product.

If the dimension n of the vector x is large, then it may be convenient to turn from problem (1) to the problem

$$\widehat{F}\hat{x}'(t) = \widehat{G}\hat{x}(t) + \hat{b}u(t), \qquad \hat{y}(t) = \langle \hat{x}(t), \hat{d}\rangle, \tag{2}$$

where the dimension of the vector $\hat{x}$ is substantially smaller. Problem (2) is called the *reduced-order* problem. Usually, the coefficients in (2) are defined by the rule

$$\widehat{F} = \Lambda F V, \qquad \widehat{G} = \Lambda G V, \qquad \hat{b} = \Lambda b, \qquad \hat{d} = V^* d,$$

where Λ and V are some matrices.

We discuss the case where the images of V and Λ^* contain the vectors $(\lambda_j F - G)^{-1} b$ and $\big((\lambda_j F - G)^{-1}\big)^* d$ and their iterations respectively. Here λ_j, $j = 1, \dots, p$, are given numbers. If the operators F and $\widehat{F}$ are invertible, then the image of the operator V may additionally contain the vectors b and $(F^{-1}GF^{-1})b$, while the image of the operator Λ^* may contain the vectors d, $\big((F^{-1}GF^{-1})\big)^* d$, and subsequent iterations, respectively. The linear spans of the iterations of the vectors $(\lambda_j F - G)^{-1} b$, $\big((\lambda_j F - G)^{-1}\big)^* d$, $(F^{-1}GF^{-1})b$, and $\big((F^{-1}GF^{-1})\big)^* d$ are called *Krylov's subspaces*. So the corresponding methods are named after this term. Their practical implementation is usually related to the names of *Lanczos* and *Arnoldi*.

We show that the impulse response of (2) can be represented in the form

$$\hat{h}(t) = \Big\langle \frac{1}{2\pi i} \int_\Gamma r_t(\lambda)(\lambda F - G)^{-1} b \, d\lambda, d \Big\rangle,$$

where

$$r_t(\lambda) = \sum_{j=1}^{p} \sum_{l=1}^{\varkappa_j + \chi_j} \frac{c_{lj}(t)}{(\lambda_j - \lambda)^l} + \sum_{l=0}^{\varkappa_0 + \chi_0 - 1} c_{l0}(t) \lambda^l$$

and r_t interpolates the function $\exp_t(\lambda) = e^{\lambda t}$ on the augmented spectrum of the pencil $\lambda \mapsto \lambda\widehat{F} - \widehat{G}$.

A special attention is given to the case where F is not invertible.

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Shilnikov Phenomenon Due to Variable Delay, by Means of the Fixed Point Index

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In 1967, Shilnikov showed that for a smooth vector field on a neighborhood of the origin in $\mathbb{R}^4$ with complex conjugate pairs of eigenvalues of the linearization in each open halfplane, at unequal distances from the imaginary axis, and with a homoclinic solution along which the intersection of stable and unstable manifolds is minimal, there are infinitely many periodic orbits close to the homoclinic loop. We present a similar result about shift dynamics close to a homoclinic loop for the simple-looking delay differential equation

$$x'(t) = -\alpha x(t - d(x_t))$$

where the only nonlinear ingredient is the state-dependent delay.

For α close to $5\pi/2$ so that the linear equation

$$y'(t) = -\alpha y(t-1)$$

is hyperbolic with 2-dimensional unstable space, we *construct* a delay functional $d : C \to (0, 2)$ — with $d(\phi) = 1$ on a neighborhood of $\phi = 0$ — so that the nonlinear equation has a solution which is homoclinic to zero, with the minimal intersection property as in Shilnikov's result, and with further regularity properties concerning the linearization of the semiflow along the homoclinic flow line in the *solution manifold* $X \subset C^1([-2, 0], \mathbb{R})$.

Shift dynamics for a translation along flow lines close to the homoclinic loop is then established by means of a general approach which employs the fixed point index and homotopies to maps in finite dimensions.

Equivariant Bifurcation: Ize Conjecture and Dynamics

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In this talk we review recent advances in equivariant bifurcation especially with respect to Ize conjecture. In the counter examples published so far (Lauterbach and Matthews, arXiv:1011.3986 [math.DS]; Lauterbach, arXiv:1210.2420 [math.DS]) we have seen rather simple dynamics since the flow is structured by a two dimensional system on the fixed point space for some isotropy subgroup. The invariant sphere theorem by M. Field reduces the dynamics in this case to the one on a circle.

There are some new counter examples which lead to 4-dimensional problems, i.e. the lowest dimensional invariant space is of dimension 4. Then the invariant sphere is three dimensional. We describe the lowest order equivariant mappings which occur in this context completely and look at some dynamical behavior. We also state some open problems in this context.

Existence, Uniqueness, and Regularity for the Kuramoto–Sakaguchi Equation with Unboundedly Supported Frequency Distribution

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The Kuramoto–Sakaguchi (or simply Kuramoto) equation is considered when the “frequency distribution”, the frequency being an independent variable in the model equation, has an *unbounded* support. This equation is a nonlinear, Fokker–Planck-type, parabolic integro-differential equation, and arises from the statistical description of the dynamical behavior of populations of infinitely many nonlinearly coupled random oscillators. The space-integral term in the equation accounts for mean-field interaction occurring among these oscillators. Existence, uniqueness, and regularity of solutions are here established, taking suitable limits in the formulation of the previously studied problem, where the aforementioned support was assumed to be bounded.

In this paper, we are concerned with certain problems involving an integrodifferential parabolic partial differential equation (PDE) of the Fokker–Planck-type. This PDE provides a model which describes a variety of phenomena, in particular self-synchronization of chemical and biological oscillations, hence in neurosciences, as well as in physical and social systems [1]. From the mathematical point of view, it is nonlinear through a space-integral term entering the drift coefficient, and describes the statistical time evolution of populations of infinitely many nonlinear random oscillators, subject to global (mean-field) coupling. One of the main features of these kinds of model equations is that they describe a rather ubiquitous phenomenon that is the transition from incoherence to synchronized states. It is noteworthy that, in practice, similar populations including only finitely many members are described very well by the limiting case of infinitely many oscillators. Numerical experiments show that even populations consisting only of few hundreds oscillators are described rather satisfactorily in this way [1].

The Kuramoto–Sakaguchi (or just Kuramoto) equation is a Fokker–Planck equation, where the space variable is an angle and the drift is quadratically nonlinear through an integral which accounts for the action of the infinitely many oscillators on any given one of them. An additional integration over all frequencies appears, which makes the model even more nonstandard from the mathematical viewpoint. It is given by

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial \theta^2} - \frac{\partial}{\partial \theta}(v\,\rho), \tag{1}$$

on the domain $\{(\theta, t) \in [0, 2\pi] \times [0, T]\}$, for some arbitrary $T > 0$, where the drift velocity, v, is

$$v \equiv v(\theta, \omega, t) := \omega + K\,r\,\sin(\psi - \theta), \quad r\,e^{i\,\psi} = \int_0^{2\pi}\int_{-\infty}^{+\infty} e^{i\,\varphi}\,\rho(\varphi, t, \omega)\,g(\omega)\,d\omega d\varphi. \tag{2}$$

Here $D > 0$ represents the diffusion coefficient, $K > 0$ the coupling strength, and ρ the transition probability density of the amplitude distribution of the oscillators. The variable ω should be picked up from the support of the "frequency distribution" $g(\omega)$.

The results available in the literature concerning parabolic equations, or even integroparabolic equations, cannot be applied to this case.

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Linear Reduction of Schlesinger Equations and Their Solutions

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We consider the nonlinear Pfaff systems

$$\mathrm{d}\,B_i(a) = -\sum_{j=1,\,j\neq i}^{n} [B_i(a),\, B_j(a)]\frac{\mathrm{d}(a_i - a_j)}{a_i - a_j} \tag{1}$$

in the space $\mathbb{C}^n$. Here $[B_i,\, B_j] = B_iB_j - B_jB_i$ denotes the commutator of matrices of size $p \times p$.

In a small neighborhood $U(a^0)$ of a point $a^0 = (a_1^0,\, a_2^0,\, \dots,\, a_n^0) \in \mathbb{C}^n$, where $a_i \neq a_j$ as $i \neq j$, equations (1) give sufficient conditions of isomonodromic deformation of the linear system of differential equations

$$\frac{\mathrm{d}\,y(z)}{\mathrm{d}\,z} = \left(\sum_{i=1}^{n} \frac{B_i}{z - a_i^0}\right) y(z) \tag{2}$$

on the Riemann sphere $\bar{\mathbb{C}}$ with $n{+}1$ singular points $a_1^0,\, \dots,\, a_n^0,\, a_{n+1}^0 = \infty$. System (1) is called the system of *Schlesinger equations*.

Paper [3] contains sufficient conditions for the upper-triangular monodromy representation of system (2) where all matrices $B_1(a), \dots, B_n(a)$ with some constant matrix C are uniformly reduced to the upper-triangular form. The system of Schlesinger equations (1) is then reduced to the nonhomogeneous linear Pfaff system in $\mathbb{C}^n$ [2, 3].

In the case where the upper-triangular matrices $B_i(a)$, $i = 1, 2, \dots, n$, (of the original system (1)) are of size $p = 2$ or $p = 3$, we give explicit integral forms for their entries. These explicit integral forms use the notion of Pochhammer loops [7] and they are generalizations in homology and cohomology with local coefficients [1,4–6].

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Fully Nonlinear Elliptic Differential Inequalities in Unbounded Domains

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We present some new a priori estimates for viscosity solutions of fully nonlinear second-order elliptic inequalities in arbitrary unbounded domains. We consider both degenerate elliptic inequalities with "absorbing" lower-order terms satisfying generalized Keller–Osserman type conditions and uniformly elliptic inequalities with reaction power-like zero order terms. When the inequalities are posed in the whole space or in cone-like domains, we give necessary and sufficient conditions for the existence of subsolutions and supersolutions.

Lyapunov Functions in Dimension Estimates of Attractors of Dynamical Systems

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Harmonic oscillations are characterized by amplitude, period, and frequency, while periodic oscillations are characterized by period. To investigate and describe more complicated oscillations, one has to introduce new numerical characteristics being various dimension type characteristics (e.g., Hausdorff dimension, fractal dimension, informational dimension) of attractors corresponding to ensembles of such oscillations. The necessity to have more than one dimension characteristic is related to the complexity of the investigated objects, i.e. strange attractors. In the three-dimensional case already, the derivation of exact formulas for the dimensions of the classical attractors (e.g., the Rössler or the Lorenz attractor) is a challenging task. Thus the problem of developing analytical methods for estimation of noninteger dimensions of fractal structure attractors arises.

First analytical results in this direction were obtained in 1980 by A. Douady and J. Oesterlé and were associated with the upper estimate of the Hausdorff dimension of attractors. Later, the Lyapunov dimension turned out to be a more convenient attractor characteristic from the computational viewpoint. Firstly, it allows one to get the upper estimate of the topological, the Hausdorff, and the fractal dimensions. Secondly, it can be regarded as a characteristic of instability of a dynamical system, and finally, it is well suited for researches using the methods of classical stability theory. The latter allowed us to introduce Lyapunov functions in the estimate of the Lyapunov dimension [1]. The idea of using Lyapunov functions in order to estimate the Lyapunov dimension was proposed by G. A. Leonov. On the basis of this idea, the exact formulas of the Lyapunov dimension of attractors for Chirikov, Henon, and Lorenz systems as well as the upper bound of the Lyapunov dimension for the Rössler attractor were obtained [2, 3].

Our report presents the brief description of the method mentioned above as well as some recent advances in the study of specific dynamical systems. In particular, we have constructed new classes of Lyapunov functions for estimating the Lyapunov dimension of attractors for various generalizations of the Lorenz system. Also the exact formulas of the Lyapunov dimension of attractors for the generalized Rössler systems have been obtained.

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On Estimates for Tensor Product of Two Ordinary Differential Polynomials with Multiple Real Zeros

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Our aim is to study the structure of the linear space $L(P)$ of minimal differential polynomials $Q(D)$ subordinated to an operator $P(D)$ in the $L^\infty(\mathbb{R}^n)$ norm, i.e., the space of operators $Q(D)$ satisfying the a priori estimate

$$\|Q(D)f\|_{L^\infty(\mathbb{R}^n)} \leqslant C_1\|P(D)f\|_{L^\infty(\mathbb{R}^n)} + C_2\|f\|_{L^\infty(\mathbb{R}^n)}$$

with constants C_1 and $C_2 > 0$ independent of $f \in C_0^\infty(\mathbb{R}^n)$.

We consider the case where the symbol $P(\xi)$ is the tensor product of two polynomials $P_1(\xi)$ and $P_2(\xi)$ in different variables,

$$P(\xi) = P_1(\xi) \otimes P_2(\xi) := P_1(\xi_1, \ldots, \xi_{p_1}, 0, \ldots, 0) \cdot P_2(0, \ldots, 0, \xi_{p_1+1}, \ldots, \xi_n).$$

In [1, 2], the space $L(P_1 \otimes P_2)$ was investigated in the case of elliptic operators $P_1(D)$ and $P_2(D)$. In [3], the space $L(P_1 \otimes P_2)$ was completely described if $P_1(D_1)$ and $P_2(D_2)$ are ordinary differential operators, and the zeros of the symbol $P_1(\xi_1)$ are all real and simple. In the above situations, the basis of $L(P)$ consists of both differential monomials and the polynomial P.

Here we consider the same problem where $P_1(D_1)$ and $P_2(D_2)$ are still ordinary differential operators whose symbols can have multiple real zeros. In this case, we construct an example of a higher-order operator P such that the space $L(P)$ contains a nontrivial linear combination of differential monomials Q_1 and Q_2 while the monomials themselves do not belong to $L(P)$.

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Mathematical modelling and simulations of ischaemic brain cell swelling

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Ischaemic brain stroke caused by occlusion of one of the cerebral arteries leads to the development of brain oedema. Consequent brain tissue swelling results in increase of intracranial pressure and in many cases leads to death. First malignant changes to the brain condition are observable at the stage of cytotoxic (cellular) oedema.

The process is characterized by swelling of brain cells that absorb water from the extracellular space.

In this talk the development of a mathematical model for the one cell swelling problem and simulation results for the obtained model will be discussed. Model equations include the Biot poroelasticity equations used to describe the behavior of the interior of the cell where the Biot system is coupled with the equations for the extracellular fluid flow, i.e. the Stokes system. As the driving force of the swelling is osmotic pressure that depends on the ionic concentration difference between extra- and intracellular spaces, advection-diffusion equations are also considered. The numerical scheme for a simplified model is based on a partitioned approach where the poroelasticity and fluid flow equations are solved independently.

About Stability of Economic System Development

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Consider an economic system S consisting of subsystems s_i, $i = \overline{1,n}$, interconnected to each other. The stability of the whole economic system is determined based on stability of its subsystems.

Let the differential equation

$$\dot{x} = f(t,x) \tag{1}$$

describe the economic system S, where $x(t) \in \mathbb{R}^n$ is the economics state, $f(t,x) \in C^{(0,1)}([T,+\infty) \times \mathbb{R}^n, \mathbb{R}^n)$ is the demand function, and we suppose that the solution of (1) exists for all initial conditions. In addition, the state of the economic system S is a nonnegative vector $x \in \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geqslant 0\}$ and we suppose also that $f(t,0) = 0$ as $t \geqslant T$, where $x = 0$ is a unique equilibrium state of the economic system S described by the differential equation (1).

Theorem 1. *The solution of differential equation* (1) *is absolutely uniformly bounded for* $\|x_0\| \leqslant r$, $t \geqslant T$, *if and only if there exist functions* $V, W : [T,+\infty) \times \mathbb{R}^n \to (0,+\infty)$ *satisfying the following conditions:*

a) $V(t,x), W(t,x) \to +\infty$ *as* $\|x\| \to +\infty$ *uniformly with respect to t;*

b) $V(t,x) \leqslant \rho_1(r)$, $W(t,x) \leqslant \rho_2(r)$ *for* $\|x_0\| \leqslant r$*;*

c) $V(t,x(t))$ *and* $W(t,x(t))$ *are correspondingly nonincreasing and nondecreasing functions, where* $x(t)$ *is the solution to* (1).

Theorem 2. *Consider the set* $E = \{x(t;t_0,x_0) : T \leqslant t, t_0 < +\infty, x_0 \in \mathbb{R}^n\}$ *uniformly bounded with respect to t and* t_0*, and absolutely uniformly bounded solutions* $x(t;t_0,x_0)$ *as* $x_0 \in \mathbb{R}^n$*. If the limits* $\lim\limits_{t\to+\infty} x(t;t_0,x_0)$ *exist and finite for all* $t_0 \geqslant T$ *and* $x_0 \in \mathbb{R}^n$*, then Eq.* (1) *has an asymptotic equilibrium.*

Thus, if the assumptions of Theorem 1 are fulfilled for the solution of differential equation (1), then by Theorem 2 the economic system described by equation (1) with interaction matrix of subsystems D has an asymptotic equilibrium.

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Riccati Equation Method for Second Order Neutral Half-Linear Equations

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We present oscillation criteria for the second-order neutral delay half-linear differential equation
$$\Big[r(t)\Phi(z'(t))\Big]' + q(t)\Phi(x(\sigma(t))) = 0,$$
where $\Phi(t) = |t|^{\alpha-1}t$, $\alpha \geqslant 1$, $z(t) = x(t) + p(t)x(\tau(t))$, p and q are positive functions. We use the method of Riccati type substitution. In order to obtain results sharper than previous results of the authors, we modify the estimates usualy used in the related literature. We construct an example of the neutral Euler-type equation to show that our results are sharp. We discuss also the case where $\sigma \circ \tau \neq \tau \circ \sigma$.

The presented results are joint work with Simona Fišnarová.

On Doubly Nonlinear Evolution Equations of Elliptic-Parabolic-Hyperbolic Type

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At the beginning of the talk, an introduction to doubly nonlinear reaction-diffusion equations
$$\frac{\partial b(u)}{\partial t} - \operatorname{div}(a(\nabla u)) = f(u) \tag{1}$$
is given and an appropriate functional analytic framework is presented. Equation (1) may be considered as a model of the filtration of a reactive non-Newtonian fluid through a porous medium. For the prototypical case $a(\nabla u) = |\nabla u|^{p-2}\nabla u$, $b(u) = |u|^{m-2}u$, quasilinear PDE (1) becomes degenerate and / or singular in dependence on the parameters $1 < m, p < \infty$, $m, p \neq 2$, and is elliptic-parabolic.

In the main part of the talk, recent results are reported for three different doubly nonlinear evolution equations of elliptic-parabolic-hyperbolic type, namely for equation (1) with an additional convection term $+\operatorname{div}(g(u))$, for the doubly nonlinear incompressible Navier-Stokes equations
$$\frac{\partial b(u)}{\partial t} + \operatorname{div}(b(u) \otimes u) = -d\pi + \operatorname{div}(a(\nabla^{\mathrm{sym}} u)) + f\,,$$
$$\operatorname{div}(u) = 0\,,$$

where u models the velocity vector field of an incompressible non-Newtonian fluid in a porous medium, and for the second-order equation

$$\frac{\partial}{\partial t} b\left(\frac{\partial u}{\partial t}\right) - \operatorname{div}\left(a\left(\nabla^{\mathrm{sym}}\frac{\partial u}{\partial t}\right)\right) - \operatorname{div}(\sigma(\nabla^{\mathrm{sym}} u)) = f$$

modeling the propagation of waves in imperfect viscoelastic materials.

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Justification of Sobolev's and Keller–Blank's Formulas in Scattering by Wedges

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We consider *nonstationary* scattering of plane waves by a wedge

$$W := \{y = (y_1, y_2) : y_1 = \rho\cos\theta, y_2 = \rho\sin\theta, \rho > 0, 0 < \theta < \phi < \pi\}.$$

Let $u_{in}(y,t) := e^{-i\omega_0(t-n_0\cdot y)} f(t - n\cdot y)$, where $t \in \mathbb{R}$, $y \in Q := \mathbb{R}^2 \setminus W$, $\omega_0 > 0$, be the incident plane wave. We assume $n_0 = (\cos\alpha, \sin\alpha)$, where (for simplicity) $\max(0, \phi - \pi/2) < \alpha < \min(\pi/2, \phi)$. In this case, $u_{in}(y, 0) = 0$, $y \in \partial Q$. For the profile, we have $f \in \mathbb{C}^\infty(\mathbb{R})$, $f(s) = 0$, $s < 0$, and $f(s) = 1$, $s > \mu$, for some $\mu > 0$. Let Q_1, Q_2 be the sides of Q. The scattering is described by means of the following mixed wave problems in Q (depending on the properties of the wedge):

$$\left\{\begin{array}{ll} \Box u(y,t) = 0, & y \in Q \\ P_l u(y,t) = 0, & y \in Q_l \end{array}\right. \Bigg|\ t > 0, \quad \left\{\begin{array}{l} u(y,0) = u_{in}(y,0), \\ \dot u(y,0) = \dot u(y,0), \end{array}\right. \Bigg|\ y \in Q, \quad l = 1,2, \qquad (1)$$

where $P_l = 1$ or $P_l = \partial_{n_l}$ for the exterior normals n_l to Q_l (DD, NN or DN-problems). Denote $\dot Q := \overline{Q} \setminus \{0\}$, $\{y\} := |y|/(1 + |y|)$, $y \in \mathbb{R}^2$.

Definition. Given $\varepsilon, N \geqslant 0$, by $\mathcal{E}_{\varepsilon,N}$ we denote the space of functions $u(t, y) \in C(\overline{Q} \times \overline{\mathbb{R}^+})$ with the finite norm

$$\|u\|_{\varepsilon,N} := \sup_{t\geqslant 0}\Big[\sup_{y\in\overline{Q}} |u(y,t)| + \sup_{y\in\dot Q}(1+t)^{-N}\{y\}^{\varepsilon}|\nabla_y u(y,t)|\Big] < \infty, \quad N \geqslant 0.$$

Let $\Phi := 2\pi - \phi$, $q := \pi/(2\Phi)$.

Theorem (see [3, 4]).

1. *There exists a unique solution* $u(y,t) \in \mathcal{E}_{1-2q,1-2q}$ *to DD, NN, DN-problems* (1) *in the case of DD and NN boundary conditions, and* $u(y,t) \in \mathcal{E}_{1-q,1-q}$ *in the case of DN boundary condition. The solution is expressed as the sum of the incident wave* u_{in}*, an optical wave* u_r *reflected by the sides of the wedge, and a wave* u_d *diffracted by the edge of the wedge. The wave* u_d *admits a simple Sommerfeld–Maljuzhinetz-type representation,*

$$u_d(y,t) = i\frac{e^{-i\omega_0 t}}{4\Phi}\int Z(\beta + i\theta)d\beta, \quad \theta \neq 2\phi - \alpha, \quad 2\pi - \alpha$$

with the Sommerfeld–Maljuzhinetz-type kernel Z depending on the boundary conditions and the wave profile f.

2. *The Limiting Amplitude Principle holds, which means that*

$$u(y,t) - e^{-i\omega_0 t}A(y) \to 0$$

as $t \to \infty$ *uniformly for* $|y| \leqslant \rho_0$*, where the limiting amplitude A is the Sommerfeld–Maljuzhinetz-type solution to the classical stationary diffraction problem of a plane wave by the wedge* [1].

We extend these results for the generalized incident wave using appropriate functional class of solutions [5]. In particular, we justify the classical Sobolev and Keller–Blank [6] solutions obtained for the pulse, i.e. for the incident wave which is the Heaviside step function. The theory uses the method of complex characteristics [2].

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Homogenization of Parabolic Systems with Periodic Coefficients in a Bounded Domain[1]

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Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain of class $C^{1,1}$. In $L_2(\mathcal{O};\mathbb{C}^n)$, we consider second-order elliptic matrix differential operators (DOs) $A_{\flat,\varepsilon}$, $\flat = D, N$, given by $b(\mathbf{D})^* g(\mathbf{x}/\varepsilon) b(\mathbf{D})$, with the Dirichlét or Neumann boundary conditions, respectively. Here $0 < \varepsilon \leqslant 1$; $g(\mathbf{x})$ is an $(m \times m)$-matrix-valued function which is bounded, uniformly positive definite and periodic with respect to some lattice Γ. Next, $b(\mathbf{D})$ is a first order DO with the symbol $b(\boldsymbol{\xi}) = \sum_{j=1}^{d} b_j \xi_j$, where b_j are constant $(m \times n)$-matrices. It is assumed that $m \geqslant n$ and $\operatorname{rank} b(\boldsymbol{\xi}) = n$, $0 \neq \boldsymbol{\xi} \in \mathbb{R}^d$ for $\flat = D$ and $0 \neq \boldsymbol{\xi} \in \mathbb{C}^d$ for $\flat = N$. The symbol $\partial^{\varepsilon}_{\boldsymbol{\nu}}$ stands for the corresponding conormal derivative.

We study homogenization in the small period limit for the solution $\mathbf{u}_{\flat,\varepsilon}(\mathbf{x},t)$, $\flat = D, N$, of the initial boundary value problem

$$\partial_t \mathbf{u}_{\flat,\varepsilon} = -A_{\flat,\varepsilon}\mathbf{u}_{\flat,\varepsilon} \text{ in } \mathcal{O}, \quad \mathbf{u}_{\flat,\varepsilon}|_{t=0} = \boldsymbol{\phi}, \quad \mathbf{u}_{D,\varepsilon}|_{\partial\mathcal{O}} = 0 \text{ or } \partial^{\varepsilon}_{\boldsymbol{\nu}}\mathbf{u}_{N,\varepsilon}|_{\partial\mathcal{O}} = 0.$$

Here $\boldsymbol{\phi} \in L_2(\mathcal{O};\mathbb{C}^n)$. The *effective problem* has the form

$$\partial_t \mathbf{u}_{\flat,0} = -A^0_{\flat}\mathbf{u}_{\flat,0} \text{ in } \mathcal{O}, \quad \mathbf{u}_{\flat,0}|_{t=0} = \boldsymbol{\phi}, \quad \mathbf{u}_{D,0}|_{\partial\mathcal{O}} = 0 \text{ or } \partial^{0}_{\boldsymbol{\nu}}\mathbf{u}_{N,0}|_{\partial\mathcal{O}} = 0.$$

The effective operator $A^0_{\flat}$, $\flat = D, N$, is given by $b(\mathbf{D})^* g^0 b(\mathbf{D})$ with the Dirichlét or Neumann boundary conditions, respectively. The constant positive *effective matrix* g^0 is defined as usual in homogenization theory.

Theorem. *There exists a number $\varepsilon_0 \in (0,1]$ depending on the domain $\mathcal{O}$ and the lattice Γ such that for $0 < \varepsilon \leqslant \varepsilon_0$ and $t > 0$ we have*

$$\|\mathbf{u}_{\flat,\varepsilon}(\cdot,t) - \mathbf{u}_{\flat,0}(\cdot,t)\|_{L_2} \leqslant C_{1,\flat} e^{-c_\flat t} \varepsilon (t+\varepsilon^2)^{-1/2} \|\boldsymbol{\phi}\|_{L_2}, \tag{1}$$

$$\|\mathbf{u}_{\flat,\varepsilon}(\cdot,t) - \mathbf{u}_{\flat,0}(\cdot,t) - \varepsilon\mathbf{v}_{\flat,\varepsilon}(\cdot,t)\|_{H^1} \leqslant C_{2,\flat} e^{-c_\flat t} \left(\varepsilon^{1/2} t^{-3/4} + \varepsilon t^{-1}\right) \|\boldsymbol{\phi}\|_{L_2},$$

$\flat = D, N$. *Here $\mathbf{v}_{\flat,\varepsilon}$ is the corresponding corrector. The positive constants $C_{1,\flat}$, $C_{2,\flat}$, and $c_\flat$ are controlled in terms of the problem data.*

Estimate (1) is order sharp for small ε and a fixed $t > 0$.

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[1]The talk is based on joint work [1] with T. A. Suslina.

Elliptic Problems and Hörmander Spaces

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The talk gives a survey of applications of Hörmander function spaces to the theory of elliptic differential equations. We consider an important class of these spaces named *the refined Sobolev scale*. This scale is defined in terms of the inner product

$$H^{s,\varphi} := \left\{ w \in \mathcal{S}'(\mathbb{R}^n) : \|w\|^2_{H^{s,\varphi}} := \int_{\mathbb{R}^n} \langle\xi\rangle^{2s} \varphi^2(\langle\xi\rangle) \, |(Fw)(\xi)|^2 \, d\xi < \infty \right\},$$

where $\langle\xi\rangle := (1 + |\xi|^2)^{1/2}$ and Fw is the Fourier transform of a distribution w. The regularity properties of distributions that form $H^{s,\varphi}$ is characterized by the number $s \in \mathbb{R}$ and the functional parameter $\varphi : [1, \infty) \to (0, \infty)$. The latter varies regularly at infinity in the Karamata sense. As an important example of φ, we can take the logarithmic function, its arbitrary iterations, their real powers, and product of these functions.

The refined Sobolev scale contains the Hilbert scale of Sobolev spaces $\{H^s = H^{s,1} : s \in \mathbb{R}\}$ and is attached to the last one by the parameter s since $H^{s+\varepsilon} \hookrightarrow H^{s,\varphi} \hookrightarrow H^{s-\varepsilon}$ for any $\varepsilon > 0$. The functional parameter φ refines the main (power) regularity given by s. Specifically, the space $H^{s,\varphi}$ is narrower (or broader) than the Sobolev space H^s provided that $\varphi(t) \to \infty$ (or $\varphi(t) \to 0$, respectively) as $t \to \infty$.

We discuss the following topics [1, 2]:

- the connection between the refined Sobolev scale and Sobolev spaces by means of interpolation with a functional parameter of pairs of Hilbert spaces;
- Hörmander spaces on a smooth closed manifold;
- properties of elliptic operators on the refined Sobolev scale over a smooth closed manifold (the Fredholm property, an a priori estimate of solutions, local increase in their regularity);
- some applications of Hörmander spaces to the spectral theory of self-adjoint elliptic operators (sufficient conditions under which the expansions with respect to the eigenfunctions of the operator converge almost everywhere or in C^k with $k \in \mathbb{Z}_+$);
- the Fredholm theory of elliptic boundary-value problems on the refined Sobolev scale.

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Riemann–Hilbert Problem to the Camassa–Holm Equation: Long-Time Dynamics of Step-Like Initial Data

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We consider the Camassa–Holm equation

$$u_t - u_{txx} + 2u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}$$

on the line in the class of functions with strictly positive momentum variable $u - u_{xx} + 1 > 0$. The initial data is a step-like function, i.e. $u(x, 0) \to 0$ as $x \to +\infty$ and $u(x, 0) \to c$ as $x \to -\infty$, where $c > 0$ is a positive real number. The goal is to reformulate the Cauchy problem as a vector Riemann–Hilbert problem in view of its further application to the study of the asymptotic behavior of the solution to the initial-value problem as $t \to \infty$. Using the steepest descent method and the so-called g-function mechanism, we deform the originally oscillatory vector Riemann–Hilbert problem to explicitly solvable model forms and show that, depending on whether the initial constant c lies in the interval $(0, 1)$, $(1, 3)$, or $(3, \infty)$, the xt-half-plane is divided into 5 sectors with qualitatively different asymptotic behavior of the solution to the initial-value problem: a soliton region, one or two regions of modulated elliptic waves, a region of a modulated hyper-elliptic wave of genus 2, and a region with fast convergent to the constant c wave.

Quadratic Glimm Functional for General Hyperbolic Systems of Conservation Laws

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We construct a new quadratic Glimm functional $\mathfrak{Q}$ for an approximate solution, obtained by the Glimm scheme [3], to the Cauchy problem associated with a general hyperbolic system of conservation laws,

$$u_t + f(u)_x = 0, \quad u = u(t, x) \in \mathbb{R}^n, \quad f : \mathbb{R}^n \to \mathbb{R}^n,$$

without any additional assumption on f besides the strict hyperbolicity.

The definition of $\mathfrak{Q}$ is based on a wave tracing algorithm, which splits each wave-front in the approximate solution into infinitesimal waves and thus it turns out to be dissimilar from the other ones already present in the literature (see for example [4]).

Differently from the other Glimm-type functionals already known (see [1, 3]), our functional bounds the total variation in time of the speed of each infinitesimal wave, thus providing, by the well-known arguments (see [2]), together with the fact that $\mathfrak{Q}$ has bounded total variation in time, the key step to obtain a sharp convergence rate of the Glimm scheme.

One of the main features of $\mathfrak{Q}$ is its non-locality in time, which requires a deep analysis of the past history of each pair of infinitesimal waves present in the approximate solution.

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Local Behavior of the Electromagnetic Field in the Vicinity of the Discontinuity Line for Dielectric Permittivity

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Local behavior of the electromagnetic field of the dielectric waveguide in the vicinity of the discontinuity line for dielectric permittivity is considered. The problem of determination of the electromagnetic field in the dielectric waveguide was reduced to the following elliptic boundary value problem for the components of electromagnetic field $\{H_x, H_y, E_z\}$ (see [1,2]):

$$\begin{array}{l} -\mathrm{graddiv} H_\perp - k^2\varepsilon H_\perp - ik\varepsilon \mathrm{rot} E_z = -\gamma^2 H_\perp, \\ -ik\mathrm{rot}\varepsilon H_\perp - \mathrm{div}\varepsilon\mathrm{grad} E_z = -\gamma^2 \varepsilon E_z. \end{array}$$

Boundary and conjugation conditions on the discontinuity line C of the coefficient ε are the following:

$$\begin{array}{ll} (\mathbf{H}\cdot\mathbf{n})|_{\partial\Omega} = 0, & E_z|_{\partial\Omega} = 0, \\ [(\mathbf{H}\cdot\mathbf{n})]|_C = 0, & [E_z]|_C = 0,\ (\mathbf{H}\times\mathbf{n})|_C = 0, \\ [\mathrm{div} H_\perp]|_C = 0, & [\varepsilon\,(\mathrm{grad} E_z + ik\,(\mathbf{H}\times\mathbf{i}_z))\cdot\mathbf{n}]|_C = 0, \end{array}$$

where $H_\perp = \{H_x, H_y\} = \{H_r, H_\varphi\}$ is the transversal component of the magnetic field, E_z is the longitudinal component of the electric field, ε is the dielectric permittivity, C is the discontinuity line of dielectric permittivity, $\mathbf{n}$ is the normal vector, γ is the propagation constant,

$$\begin{array}{l} \mathrm{div} H_\perp = \dfrac{\partial H_x}{\partial x} + \dfrac{\partial H_y}{\partial y}, \quad \mathrm{rot} H_\perp = \dfrac{\partial H_y}{\partial x} - \dfrac{\partial H_x}{\partial y}, \\ \mathrm{grad} E_z = \mathbf{i}_x \dfrac{\partial E_z}{\partial x} + \mathbf{i}_y \dfrac{\partial E_z}{\partial y}, \quad \mathrm{rot} E_z = \mathbf{i}_x \dfrac{\partial E_z}{\partial y} - \mathbf{i}_y \dfrac{\partial E_z}{\partial x}. \end{array}$$

The discontinuity line C of permittivity ε corresponds to rays C_1 and C_2 outgoing from the origin at the angle ω_0. All the results of the paper can be easily extended to the case where the discontinuity line consists of several rays outgoing from the origin at arbitrary angles. Asymptotic representation of the solution in the vicinity of irregularity is constructed by the method first proposed by V. A. Kondrat'ev [3,4].

The following representation for the component E_z of the electric field was obtained:

$$E_z(r,\varphi) = \chi \sum_{-\delta<\nu_k^{(1)}<1} r^{\nu_k^{(1)}} \left\{ C_k^{(1)} \cos\left[(\pi-\varphi)\nu_k^{(1)}\right] + D_k^{(1)} \cos\left[(\pi - |\omega_0-\varphi|)\,\nu_k^{(1)}\right]\right\} +$$
$$+ \chi \sum_{-\delta<\nu_k^{(2)}<1} r^{\nu_k^{(2)}} \left\{ C_k^{(2)} \cos\left[(\pi-\varphi)\nu_k^{(2)}\right] + D_k^{(2)} \cos\left[(\pi - |\omega_0-\varphi|)\,\nu_k^{(2)}\right]\right\} + \Re(r,\varphi).$$

where $\nu_k^{(1)}$ and $\nu_k^{(2)}$ are the solutions to the equations

$$\sin\pi\nu_k - \alpha\sin(\pi\nu_k - \nu_k\omega_0) = 0 \text{ and } \sin\pi\nu_k + \alpha\sin(\pi\nu_k - \nu_k\omega_0) = 0,$$

respectively (except $\nu = 0$), $C_k^{(j)}$ and $D_k^{(j)}$ $(j = 1,2)$ are constants, $\delta > 0$ is a sufficiently small number, $\Re(r,\varphi)$ is the smooth part of solution, $\chi(r)$ is a patch function,

$$\chi(r) = \begin{cases} 1, & r \leqslant d/2, \\ 0, & r > d, \end{cases} \qquad \chi(r) \in C^\infty.$$

The magnetic field has a singularity of lower degree than the electric field.

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Singular Solutions of Strongly Elliptic Differential-Difference Equations in Half-Plane

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Consider the equation

$$u_{xx} + a u_{xx}(x+h,y) + u_{yy} = 0,\ x \in (-\infty,+\infty),\ y \in (0,+\infty),$$

with real parameters a and h.

To guarantee the strong ellipticity of the equation, we assume that $|a| < 1$.

Denote $\sqrt{2a\cos h\xi + a^2 + 1}$ by $\varphi(\xi)$ and consider the function

$$\mathcal{E}(x,y) = \int_0^\infty e^{-yG_1(\xi)} \cos\left[x\xi - yG_2(\xi)\right] d\xi,$$

where $G_1(\xi) = \xi\sqrt{\dfrac{\varphi(\xi) + a\cos h\xi + 1}{2}}$ and $G_2(\xi) = \xi\sqrt{\dfrac{\varphi(\xi) - a\cos h\xi - 1}{2}}$.

The presented result is as follows: *the function $\mathcal{E}(x, y)$ is well defined in the half-plane $(-\infty, +\infty) \times (0, +\infty)$ and its convolution with any bounded and continuous on the real axis function is a classical solution of the considered equation.*

Two Fractional Laplacians

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Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $s > 0$ a given real number. For any sufficiently regular function u on $\mathbb{R}^n$ such that $u \equiv 0$ outside Ω, one can consider the "Dirichlét" fractional Laplacian $(-\Delta)^s_D u$ given via the Fourier transform by

$$\mathcal{F}\left[(-\Delta)^s_D u\right](\xi) = |\xi|^{2s}\mathcal{F}[u](\xi),$$

and the "Navier" fractional Laplacian

$$(-\Delta_\Omega)^s_N u = \sum\nolimits_j \Big(\lambda_j^s \int\limits_\Omega u\varphi_j\Big)\varphi_j.$$

Here λ_j and φ_j are the eigenvalues and the eigenfunctions of the Dirichlét Laplacian in Ω, respectively.

We compare these two fractional Laplacians when $s \in (0, 1)$. In particular, the following facts are proved:

1. *The domains of the forms $\langle(-\Delta)^s_D u, u\rangle$ and $\langle(-\Delta)^s_N u, u\rangle$ coincide with*

$$\widetilde{H}^s(\Omega) = \left\{u \in H^s(\mathbb{R}^n) \mid supp(u) \subseteq \overline{\Omega}\ \right\}.$$

2. *The difference operator $(-\Delta)^s_N - (-\Delta)^s_D$ is positive definite and positive preserving.*

3. *For any fixed $u \in \widetilde{H}^s(\Omega)$, one has*

$$\langle(-\Delta)^s_D u, u\rangle = \inf_{\Omega' \supset \Omega}\langle(-\Delta)^s_N u, u\rangle$$

   (*the infimum is taken over the family of smooth bounded domains*).

4. *Assume $n \geqslant 2$ or $s < 1/2$, and put $2^*_s = \frac{2n}{n-2s}$. Then the "Dirichlét–Sobolev" and the "Navier–Sobolev" constants coincide, i.e.*

$$\inf_{\substack{u\in\widetilde{H}^s(\Omega)\\ u\neq 0}} \frac{\langle(-\Delta)^s_D u, u\rangle}{\|u\|^2_{L^{2^*_s}(\Omega)}} = \inf_{\substack{u\in\widetilde{H}^s(\Omega)\\ u\neq 0}} \frac{\langle(-\Delta)^s_N u, u\rangle}{\|u\|^2_{L^{2^*_s}(\Omega)}}.$$

This talk is based on a joint paper with Alexander I. Nazarov, see [1].

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The Brezis–Nirenberg Effect for Fractional Laplacians

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Let m and s be given real numbers obeying the inequalities $0 \leqslant s < m < n/2$, and $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Denote by $2^*_m = \frac{2n}{n-2m}$ the critical Sobolev exponent for the embedding $W_2^m \hookrightarrow L_q$.

We study the equations

$$(-\Delta)^m u = \lambda(-\Delta)^s u + |u|^{2^*_m - 2} u \qquad \text{in} \quad \Omega \quad \text{and} \tag{1}$$

$$(-\Delta)^m u = \lambda |x|^{-2s} u + |u|^{2^*_m - 2} u \qquad \text{in} \quad \Omega \tag{2}$$

with the Dirichlét boundary conditions understood in a proper way. Namely, the fractional Dirichlet–Laplacian $(-\Delta)^m$ is a self-adjoint operator defined by its quadratic form by putting

$$\int\limits_{\mathbb{R}^n} (-\Delta)^m u \cdot u \, dx := \int\limits_{\mathbb{R}^n} |\xi|^{2m} |\mathcal{F}[u]|^2 \, d\xi \,, \qquad u \in \widetilde{H}^m(\Omega),$$

where $\widetilde{H}^m(\Omega) = \{u \in W_2^m(\mathbb{R}^n) \, : \, \operatorname{supp} u \subset \overline{\Omega}\}$ and $\mathcal{F}$ stands for the Fourier transform. Dealing with (2), we always assume that Ω contains the origin.

Theorem 1. *Let $s \geqslant 2m - \frac{n}{2}$. Then each of problems* (1) *and* (2) *has a nontrivial weak solution in $\widetilde{H}^m(\Omega)$.*

The case $s = 0$ and m integer or $m \in (0, 1)$ was considered earlier in a number of papers beginning with the celebrated paper [1] (for $m = 1$). We cite also [2], where equation (1) was studied in the case $m = 2$, $s = 1$.

This talk is based on a joint paper with Roberta Musina, see [3]. Author was supported by RFBR grant 14-01-00534 and by Saint Petersburg State University grant 6.38.670.2013.

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On Geometrical Mechanism of Destruction of Adiabatic Invariance in Slow-Fast Hamiltonian Systems

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We discuss a geometrical mechanism of destruction of adiabatic invariance in slow-fast Hamiltonian systems with separatrices in the phase space. In contrast to a slow diffusive-like destruction typical for many systems, this mechanism is responsible for very fast loss of adiabatic invariance in a large phase volume. The main effect is due to a geometrical asymmetry of separatrices and corresponding geometrical jumps of an adiabatic invariant.

We consider a slow-fast Hamiltonian system with two degrees of freedom. The Hamiltonian of this system has the form $H = H(p, q, y, x)$, where pairs of conjugate variables are (p, q) (fast variables) and $(y, \varepsilon^{-1}x)$ (thus (y, x) are slow variables); $0 < \varepsilon \ll 1$. Dynamics of fast variables for frozen values of slow variables is called a fast motion. We assume that the phase portrait of the fast motion is divided by separatrices into domains filled by closed trajectories. Dynamics of slow variables is approximately described by the system averaged over the fast motion. For a fixed value of the Hamiltonian, the phase space of the averaged system is obtained from the level surface of the Hamiltonian by identifying points on the same trajectory of the fast motion. This factorisation results in a 2D singular surface that branches on a curve corresponding to separatrices of the fast motion (so called uncertainty curve). The averaged system has a first integral in each smooth part of its phase space. This first integral is the action variable of the fast motion. It is an approximate first integral (an adiabatic invariant) of the exact system. Prolongation of a trajectory of the averaged system through the uncertainty curve by continuity leads to a jump in the value of action. This jump is related to difference in form of trajectories of the fast motion near the separatrices in different domains and is called a geometrical jump. Multiple passages through an uncertainty curve may lead to complete destruction of adiabatic invariance. Mathematical study of this mechanism of destruction of adiabatic invariance has been started only recently. In this talk we discuss examples of manifestation of this mechanism in problems of motion of charged particles.

Regularity of Solutions to the Second (Third) Boundary-Value Problem for Differential-Difference Equations

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The second and third boundary-value problems for second-order differential-difference equations with variable coefficients on a finite interval $(0, d)$ are considered. The following question is studied: Under what conditions will the boundary-value problem for a differential-difference equation have a classical solution for an arbitrary continuous right-hand side? It is proved that a necessary and sufficient condition for the existence of a classical solution is that certain coefficients of the difference operators on the orbits generated by the shifts be equal to zero. In the contrast to the first boundary-value problem [1], these conditions do not coincide [2].

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Inverse Limits and Chaotic Attractors

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In 1990, Barge and Martin presented a method of construction of global attractors for planar homeomorphisms in terms of inverse limits. This technique can also be extended to obtain attractors arising as inverse limits of degree one maps of the circle. That way we can obtain attractors with very strange topological structure, such as pseudoarcs or pseudocircles. In this talk, we are going to survey some known results on dynamics on various types of continua that can be obtained as attractors. We are also going to mention some examples of maps in these spaces that cannot be constructed as shift homeomorphisms on inverse limit and present a few open problems that arise.

At the end, we are going to present recent results obtained jointly with Jan Boroński. Among others, we are going to explain how to obtain a pseudocircle as an attractor of a map on a torus with non-unique rotation vectors on it. While it does not solve the Franks–Misiurewicz conjecture, it provides another method of construction of such attractors (the classical example of this type is the so-called Birkhoff attractor).

Identifying Functions Relating to the Ulam Stability Problem

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In the talk, we deal with a new identifying problem for linear functional operators of the kind $\mathcal{P}F = \sum c_j F \circ a_j$, which significantly generalizes the well-known Ulam stability problem. The obtained results prove to be very useful when processing experimental data of any kind as they enable to determine with high precision the structure of a compactly supported Banach-valued function F by using a rather restricted information concerning $\mathcal{P}F$.

On Decay of Almost Periodic Entropy Solutions to Multidimensional Scalar Conservation Laws

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The Cauchy problem

$$u_t + \mathrm{div}_x \varphi(u) = 0, \quad u(0,x) = u_0(x) \tag{1}$$

is considered in the half-space $\Pi = (0, +\infty) \times \mathbb{R}^n$.

We assume that the flux vector $\varphi(u)$ is merely continuous while the initial function $u_0(x) \in \mathfrak{B}^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is a bounded Besicovitch almost periodic function. It is known [1] that there exists a Kruzhkov entropy solution $u(t,x) \in L^\infty(\Pi)$ of problem (1). Denote by $\mathfrak{M} \subset \mathbb{R}^n$ the smallest additive subgroup containing the spectrum of u_0. We show that for a.e. $t > 0$ the function $u(t,\cdot)$ belongs $\mathfrak{B}^1(\mathbb{R}^n)$ and its spectrum is contained in $\mathfrak{M}$. Moreover, $u(t,\cdot)$ is unique as an element of $\mathfrak{B}^1(\mathbb{R}^n)$. Now suppose that the following genuine nonlinearity condition holds:

$$\forall \xi \in \mathfrak{M} \text{ the function } u \to \xi \cdot \varphi(u) \text{ is not linear on nonempty intervals.} \tag{2}$$

Denote by C_R a cube $\{\, x \in \mathbb{R}^n \mid |x_i| \leqslant R/2,\ i = 1, \dots, n\}$. Under condition (2), we establish the following decay property of the solution $u(t,\cdot)$ as $t \to +\infty$.

Theorem 1. *Let* $c = \lim\limits_{R\to+\infty} R^{-n} \int_{C_R} u_0(x)dx$. *Then*

$$\operatorname*{ess\,lim}_{t\to+\infty} \lim_{R\to+\infty} R^{-n} \int_{C_R} |u(t,x) - c| dx = 0.$$

In the case of periodic initial data, this assertion was proved in [2].

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Simulating Spatial Dynamics with IBM/ABM and Modelling Non-Local Interactions in Geo-Environmental Systems

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Modelling and simulating spatial dynamics has gained momentum in earth sciences over the last years, describing and predicting the behaviors of ecological systems, geographical populations changes in space and time, forest fires, epidemics etc.

A wide range of models and algorithmic simulations has been produced, in continuous or discrete space and time, with varying degree of success. These range from classical dynamical systems models to integro-difference models and from cellular automata to individual-based and agent-based simulations (IBM/ABM).

The international scientific literature is already enriched enough with such new models and simulations, but it seems that deadlocks have already appeared in some areas. Indeed, diverging trends have emerged and, despite the significant progress made in this field, the literature is short of a comprehensive evaluation of what has been achieved, of what remains unanswered and why it does so.

Hence, it is time to assess the achievements made so far, to consider the problems outstanding and to propose possible solutions. This constitutes the main effort in this study.

The method adopted consists in an attempt for a critical evaluation of the scientific literature that has been published over the last years.

The results reveal some main fields of research that may attract attention in the future: i.e. individual-based / agent-based simulations (IBM/ABM), non-local interaction models derived from differential, integral, and integro-difference equations.

Concluding, it appears that such completely different approaches offer complementary insights into the development and evolution of geo-environmental systems. Examples of this complementarity in geo-environmental systems can be sought in explaining self-organization, emergence, stability, non-local interactions and enhancing predictability.

On Gevrey Orders of Power Series Solutions to Third, Fourth, and Fifth Painlevé Equations

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We consider the fifth Painlevé equation. For all values of its complex parameters, we are looking for solutions in the form $\sum\limits_{s \in \mathbf{K}} c_s(z) z^s$, where $\mathbf{K}$ is a countable set and $c_s(z)$ are either complex constants, or polynomials, or series in $\log z$ (see [2, 3]).

We analyze the obtained power expansions and calculate the Gevrey order [1] of each power expansion solving the fifth Painlevé equation [5]. We also calculate the Gevrey orders of power expansions obtained for the third [5] and the fourth [6] Painlevé equations.

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On a Long Range Segregation Problem

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Segregation phenomena occurs in many areas of mathematics and science: from equipartition problems in geometry, to social and biological process (cells, bacteria, ants, mammals) to finance (sellers and buyers). There is a large body of literature studying segregation models where the interaction between species is punctual. There are many processes though, where the growth of a population at a point is inhibited by the populations in a full area surrounding that point. The work we present is a first attempt to study the properties of such a segregation process.

This is a joint paper with Luis Caffarelli and Veronica Quitalo.

Oscillatory and Nonoscillatory Solutions for Difference Equations with Several Delays[1]

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The aim of this work is to study the existence of oscillatory and nonoscillatory solutions of the linear difference equation with several delays

$$\Delta u\,(k) + \sum_{i=1}^{n} p_i\,(k)\,u\,(\tau_i\,(k)) = 0, \tag{1}$$

where $\Delta u\,(k) = u\,(k+1) - u\,(k)$ and

$$p_i : \mathbb{N} \to \mathbb{R}^+, \;\; \tau_i : \mathbb{N} \to \mathbb{N}, \tag{2}$$

$$\tau_i\,(k) \leqslant k - 1 \text{ for } k \in \mathbb{N} \text{ and } \lim_{k\to+\infty} \tau_i\,(k) = +\infty \tag{3}$$

as $1 \leqslant i \leqslant n$.

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Boundary Value Problems for Compressible Navier–Stokes Equations

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We study the initial boundary value problem for the compressible Navier-Stokes equations describing the barotropic motion of a viscous gas. It is assumed that the gas occupies a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. The state of the gas is characterized by the velocity field $\mathbf{u}(x,t)$ and the density $\varrho(x,t) \geqslant 0$. It is assumed that the gas pressure p equals ϱ^γ, where we have $\gamma \geqslant 1$ for the adiabatic exponent γ. Under these assumptions, the motion of the gas is described by the equations

$$\partial_t(\varrho\mathbf{u}) + \text{div } (\varrho\mathbf{u} \otimes \mathbf{u}) + \nabla\varrho = \text{div } \mathbb{S}(\mathbf{u}) + \varrho\mathbf{f} \quad \text{in} \quad \Omega \times (0,T), \tag{1}$$

$$\partial_t\varrho + \text{div } (\varrho\mathbf{u}) = 0 \quad \text{in} \quad \Omega \times (0,T), \tag{2}$$

where $\mathbf{f}$ is a given field of mass forces, and the viscous stress tensor is given by

$$\mathbb{S}(\mathbf{u}) = \nu_1\big(\nabla\mathbf{u} + \nabla\mathbf{u}^\top\big) + \nu_2 \text{div } \mathbf{u}\,\mathbb{I}, \quad \nu_1 > 0, \nu_1 + \nu_2 \geqslant 0. \tag{3}$$

These equations are supplemented with the boundary and initial conditions

$$\begin{aligned} \mathbf{u} = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \\ \mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \varrho(x,0) = \varrho_0(x) > 0 \quad \text{in} \quad \Omega. \end{aligned} \tag{4}$$

It is well known, see P. L. Lions, E. Feireisl, A. Novotny, I. Straskraba, that the problem has a weak solution if the adiabatic exponent γ is greater than some critical value $\gamma = \gamma_c$. The critical value γ_c equals 1 when $d = 2$ and equals $3/2$ when $d = 3$. In this work, we are focused on the limiting case $\gamma = \gamma_c$. For $d = 2$ and $\gamma = 1$, we prove the solvability of problem (1)–(4) and derive L_p-estimates for the density ϱ. For $d = 3$ and $\gamma = 3/2$, we investigate in details the compactness properties of the energy tensor.

Homogenization of the Initial-Boundary Value Problem in Perforated Domain for Parabolic Equation with p-Laplace Operator and Nonlinear Robin-Type Boundary Conditions

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The talk focuses on the study of the asymptotic behavior as $\varepsilon \to 0$ of the solution u_ε to the initial-boundary value problem for the nonlinear equation

$$\partial_t u_\varepsilon - \Delta_p u_\varepsilon \equiv \partial_t u_\varepsilon - \text{div}(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon) = f,$$

where $p \in [2, n)$, in an ε-periodically perforated domain $\Omega_\varepsilon \subset \mathbb{R}^n$, $n \geqslant 3$, with the nonlinear boundary condition of third type $\partial_{\nu_p} u_\varepsilon + \varepsilon^{-\gamma}\sigma(x, u_\varepsilon) = \varepsilon^{-\gamma} g(x)$ specified on the boundaries of the holes, where $\partial_{\nu_p} u_\varepsilon \equiv |\nabla u_\varepsilon|^{p-2}(\nabla u_\varepsilon, \nu)$ and ν is the outward unit normal vector on the boundary of a hole. It is assumed that the diameter of a hole is equal to $C_0\varepsilon^\alpha$, where $C_0 > 0$, $1 < \alpha < \frac{n}{n-p}$, and $\gamma = \alpha(n-1) - n$.

Let $\Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon}$, $S_\varepsilon = \partial G_\varepsilon$, $\partial\Omega_\varepsilon = \partial\Omega \bigcup S_\varepsilon$. In the cylinder $Q_\varepsilon^T = \Omega_\varepsilon \times (0, T)$, consider the problem

$$\begin{cases} \partial_t u_\varepsilon - \Delta_p u_\varepsilon = f(x,t), & (x,t) \in Q_\varepsilon^T, \\ \partial_{\nu_p} u_\varepsilon + \varepsilon^{-\gamma}\sigma(x, u_\varepsilon) = \varepsilon^{-\gamma} g(x), & (x,t) \in S_\varepsilon^T = S_\varepsilon \times (0,T), \\ u_\varepsilon = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u_\varepsilon(x,0) = 0, & x \in \Omega_\varepsilon. \end{cases} \tag{1}$$

It is assumed that $f \in L_2(Q_\varepsilon^T)$ and $g \in C(\overline{\Omega})$, $\sigma(x,u)$ is a continuously differentiable function of $x \in \overline{\Omega}$ and $u \in \mathbb{R}$ such that $\sigma(x,0) = 0$ and there are positive constants k_1 and k_2 satisfying the inequalities $(\sigma(x,u) - \sigma(x,v))(u-v) \geqslant k_1|u-v|^p$ and $|\sigma(x,u)| \leqslant k_2|u|^{p-1}$.

A solution of the problem is a function u_ε from the class $L_p(0,T; W^{1,p}(\Omega_\varepsilon, \partial\Omega))$ such that $\partial_t u_\varepsilon \in L_q(0,T; W^{-1,q}(\Omega_\varepsilon, \partial\Omega))$ and $u_\varepsilon(x,0) = 0$.

Passing to the scale limit, we derive the effective equations for the problem under consideration. If $\alpha \in (1, \frac{n}{n-p}]$, $\gamma = \alpha(n-1) - n$, then adsorption process on the inclusions at the micro-scale gives rise to an effective sink/source term in the macroscopic equation. The critical case $\alpha = \frac{n}{n-p}$, $\gamma = \frac{n}{n-p}(p-1)$ was studied in [1] where a homogenized problem with a new nonlinear term was constructed. For the non-critical case, the following homogenization theorem is proved, stating that the homogenized problem contains the same nonlinear term as the original one.

Theorem 1. *Let $n \geqslant 3$, $2 \leqslant p < n$, $1 < \alpha < n/(n-p)$, $\gamma = \alpha(n-1) - n$, and u_ε be a weak solution of problem* (1). *Let $u \in L_p(0,T; W_0^{1,p}(\Omega))$ with $\partial_t u \in L_q(0,T; W^{-1,q}(\Omega))$ be a weak solution of the problem*

$$\begin{cases} \partial_t u - \Delta_p u + \mathcal{A}(\sigma(x,u) - g(x)) = f(x,t), & (x,t) \in \Omega^T = \Omega \times (0,T), \\ u = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = 0 & x \in \Omega, \end{cases} \tag{2}$$

where $\mathcal{A} = C_0^{n-1} \cdot \omega_n$ and ω_n is the surface area of the unit sphere in $\mathbb{R}^n$. Then $\widetilde{u}_\varepsilon \rightharpoonup u$ in $L_2(0,T; W_0^{1,p}(\Omega))$ as $\varepsilon \to 0$, where $\widetilde{u}_\varepsilon$ is an extension of u_ε to the cylinder Ω^T.

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Spectral Properties of Nonself-Adjoint Differential Operator of Fourth Order

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Let $\mathrm{L}_2[\mathbf{a},\mathbf{b}]$ be the Hilbert space of all measurable complex-valued functions on $[\mathbf{a},\mathbf{b}]$ with the inner product $(x,y)=\int_{\mathbf{a}}^{\mathbf{b}} x(\tau)\overline{y(\tau)}\,\mathrm{d}\tau$, $x,\ y\in \mathrm{L}_2[\mathbf{a},\mathbf{b}]$. By $W_2^4[\mathbf{a},\mathbf{b}]$, denote the Sobolev space $\{y:[\mathbf{a},\mathbf{b}]\to\mathbb{C}: y,y',y''$ are continuously differentiable on $[\mathbf{a},\mathbf{b}]$, y''' is absolutely continuous on $[\mathbf{a},\mathbf{b}]$ and $y^{IV}\in \mathrm{L}_2[\mathbf{a},\mathbf{b}]\}$.

We consider two operators $L_i:\mathrm{D}(L_i)\subset \mathrm{L}_2[\mathbf{a},\mathbf{b}]\to \mathrm{L}_2[\mathbf{a},\mathbf{b}], i=1,2$, determined by the following differential expressions:

$$l(y)=y^{IV}-a(t)y''-b(t)y,\quad \text{where}\quad a,b\in \mathrm{L}_2[\mathbf{a},\mathbf{b}].$$

Here $\mathbf{a}=0$, $\mathbf{b}=1$ for operator L_1 and $\mathbf{a}=-1$, $\mathbf{b}=1$ for operator L_2. The domain $D(L_i)$, $i=1,2$, is given by the following boundary conditions:

$(bc)_1$ $y(0)=y(1)=0$, $y''(0)=y''(1)=0$ for the operator L_1,

$(bc)_2$ $y(-1)=y(1)=0$, $y'(-1)=y'(1)=0$ for the operator L_2.

Hence, $\mathrm{D}(L_i)=\{y\in W_2^4[\mathbf{a},\ \mathbf{b}]: y$ satisfies $(bc)_i\}$, $i=1,2$. The operators $L_{0i}:\mathrm{D}(L_{0i})=\mathrm{D}(L_i)\subset \mathrm{L}_2[\mathbf{a},\mathbf{b}]\to \mathrm{L}_2[\mathbf{a},\mathbf{b}]$, $L_{0i}y=y^{IV}$, $i=1,2$, are self-adjoint positive operators with compact resolvent.

The spectra $\sigma(L_{0i})$, $i=1,2$, have the form

$(bc)_1$: $\sigma(L_{01})=\{\lambda_1,\lambda_2,\dots\}$, where $\lambda_n=\pi^4n^4$, $n\in\mathbb{N}$. The corresponding eigenfunctions are $e_n(t)=\sqrt{2}\sin\pi nt$, $n\in\mathbb{N}$.

$(bc)_2$: $\sigma(L_{02})=\{\lambda_1,\lambda_2,\dots\}$, where $\lambda_n=\mu_n^4$, $\mu_n=-\frac{\pi}{4}+\pi n$ if $n\in\mathbb{N}$ is an even number, and $\mu_n=\frac{\pi}{4}+\pi n$ if n is an odd number. The corresponding eigenfunctions are

$$e_n(t)=\frac{1}{\alpha_n}(\cos\mu_n\mathrm{ch}(\mu_nt)-\mathrm{ch}\mu_n\cos(\mu_nt))\quad\text{for an odd } n,$$

$$e_n(t)=\frac{1}{\beta_n}(\sin\mu_n\mathrm{sh}(\mu_nt)-\mathrm{sh}\mu_n\sin(\mu_nt))\quad\text{for an even } n.$$

The Riesz projectors P_n are defined as $P_nx=(x,e_n)e_n$ for all $x\in \mathrm{L}_2[\mathbf{a},\mathbf{b}]$ and $n\in\mathbb{N}$.

Using the Similar Operator Method, we obtain the following results.

Theorem 1. *The differential operators L_1 and L_2 are operators with compact resolvent. The eigenvalues $\widetilde{\lambda}_n$ of the operator L_1 have the asymptotic*

$$\widetilde{\lambda}_n=(\pi n)^4+(\pi n)^2\int_0^1 a(t)\,\mathrm{d}t-(\pi n)^2\int_0^1 a(t)\cos 2\pi nt\,\mathrm{d}t-n^2\sum_{\substack{l=1\\ l\neq n}}^{\infty}\frac{a_{nl}a_{ln}l^2}{l^4-n^4}+\gamma_nn^2,$$

where (γ_n) is a summable sequence and

$$a_{nl}=\int_0^1 a(t)\cos\pi(l-n)t\,\mathrm{d}t-\int_0^1 a(t)\cos\pi(l+n)t\,\mathrm{d}t,\quad n,l\geqslant 1.$$

The analogous asymptotics holds for the operator L_2.

Theorem 2. *The operator $-L_i$, $i = 1, 2$, is a generator of an analytic semigroup.*

Also, we get the estimates of equiconvergence for spectral decompositions.

Elliptic Functional Differential Equations with Variable Coefficients and Degeneration

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Keldysh showed that elliptic differential equations with degeneration may be well posed even if one imposes no boundary conditions on a part of the boundary. There is a similar phenomenon in the theory of elliptic functional differential equations with degeneration. Elliptic functional differential equations with degeneration were studied by Skubachevskii in the case where the operator is a composition of a strongly elliptic differential operator and a degenerate differential operator. He proved that such problems can be reduced to nonlocal boundary value problems, which have applications to the theory of plasma.

We consider elliptic functional differential equations with degeneration in the case where the operator cannot be represented as a composition of a strongly elliptic differential operator and a degenerate difference operator but contains some degenerate difference operators and variable coefficients. The main difficulties here are due to the fact that the functional differential operator contains degenerate difference operators. This leads to the fact that the generalized solution need not belong to the corresponding Sobolev space, while the functional differential operator has the zero eigenvalue of infinite multiplicity.

We obtain a priori estimates of solutions of elliptic functional differential equations with degeneration. Using these estimates, we can study the spectrum of the problem and smoothness of generalized solutions in subdomains.

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On the Solvability of Operator Equations with Nonlinear Fredholm Maps of Positive Index

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The talk is devoted to solvability of some operator equations in Banach spaces with nonlinear Fredholm maps of index one. The theorems is based on the construction and properties of the topological degree with values in the bordism group of GL_c-framed q-dimensional manifolds for the class of Fredholm C^1-maps of positive index and its compact perturbations.

Theorem 1. *Let E' and E be Banach spaces, $B'(0) \subset E'$ be an open ball. Suppose $F\colon \bar{B}'(0) \subset E' \to E$ is a continuous proper map such that $F|_{B'(0)}$ is a local C^1-diffeomorphism. Let $B(0)$ be a ball in the Euclidean space $\mathbb{R}^{n+1}$ and $\Sigma h\colon B(0) \subset$*

$\mathbb{R}^{n+1} \to \mathbb{R}^n$ *be the suspension over the Hopf map*

$$h(\alpha_1, \beta_1, \alpha_2, \beta_2) = \left(2(\alpha_1\beta_1 + \alpha_2\beta_2), 2(\alpha_2\beta_1 - \alpha_1\beta_2), \alpha_1^2 + \alpha_2^2 - \beta_1^2 - \beta_2^2\right).$$

Suppose that the nonlinear Fredholm map

$$\Psi\colon D \subset E' \times \mathbb{R}^{n+1} \to E \times \mathbb{R}^n, \quad \Psi(x_1, x_2) = (F(x_1), \Sigma h(x_2) - c),$$

where $D = B'(0) \times B(0)$, $x_1 \in E'$, $x_2 \in \mathbb{R}^{n+1}$ *is such that* $\Psi(x) \neq 0$ *for* $x \in \partial D$.

Then for every vector y *belonging to the same connected component of the set* $(E \times \mathbb{R}^n) \setminus \Psi(\partial D)$ *as zero, the equation* $\Psi(x) = y$ *has a solution* $x \in D$.

Remark. If $G\colon B(0) \times [0,1] \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$ is a homotopy such that

$$G(x, 0) = \Sigma h(x) - c, \quad G(x,t) \neq 0 \quad for \quad x \in \partial B(0), \quad t \in [0,1],$$

then the equation $\Psi(x_1, x_2) = (F(x_1), G(x_2, 1)) = y$ has a solution.

Theorem 2. *Let* E' *and* E *be Banach spaces,* $D \subset E'$ *a simply connected domain,* $L\colon E' \to E$ *a linear Fredholm operator,* $\operatorname{ind} L = 1$, *and* $k\colon \bar{D} \subset E' \to E$ *a continuous compact map. Suppose that* $E_1 \subset E$ *is a subspace,* $\dim E_1 < \infty$, *and* $\mathrm{Im} L + E_1 = E$. *Let* $E_1' = L^{-1}(E_1)$ *and* $k(E_1' \cap \bar{D}) \subset E_1$. *Suppose that* $(L+k)(x) \neq 0$, $x \in \partial D$, *and the restriction of the map* $(L+k)$ *to the closure of the set* $B_1 = E_1' \cap D$ *is homotopic to the suspension over the Hopf map with a homotopy nonvanishing on* ∂B_1. *Then the equation* $(L+k)x = 0$ *has a solution.*

The theorems are applied to the problem of solvability of a system of ordinary differential equations with Hopf boundary conditions and to the bifurcation of solutions of the elliptic boundary value problem with a complex parameter.

On Suppression of Distortions in Nonlocal Models of Adaptive Optics

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Suppression of phase distortions by means of nonlocal feedback optical systems is one of the promising directions of modern adaptive optics. Experimental studies of different configurations that utilize both light phase modulators with the TV feedback loop and purely optical feedback circuit have shown their effectiveness for high-resolution distortion compensation. We have to consider the delay of a control feedback signal to provide most adequate study of real adaptive systems, namely, ones with the TV feedback loop. Furthermore, the delay may act as an additional control parameter capable of significantly enriching the spatio-temporal dynamics, which is important for modeling generators of artificial optical turbulence. Superposition of delay and rotation of spatial argument in the feedback loop leads to excitation of one-dimensional [1] and two-dimensional [2] stable rotating waves. Generally, these phenomena negatively affect the quality of distortion suppression. In this context, theoretical and numerical study of various simplified distortion suppression models is of great importance.

In this paper, we present the results of the analytical and numerical study of distortion compensation in the model of a nonlocal feedback optical system governed, as in [1], by a parabolic functional differential equation (FDE) with temporal delay and rotation of spatial argument. We propose a bi-modal model to conduct an analytical study of the influence of delay and rotation of spatial arguments on suppression quality for both stationary and dynamic harmonic distortion, and to compare it with the direct numerical simulation of FDE. The results of suppression are further compared with those ones obtained for another optical configuration governed by the Volterra-type parabolic FDE.

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Controlling the Controller: Hysteresis-Delay Differential Equations

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Our talk revolves around differential equations with hysteresis and delay terms. We focus on the problem of stability analysis of periodic solutions of such equations. This problem is infinite-dimensional and discontinuous due to the delay and hysteresis. We present a technique to reduce it, in certain cases, to the spectral problem for a linear finite-dimensional operator.

Our main application is a thermal control model. It consists of a parabolic equation with hysteresis on the boundary. Gurevich and Tikhomirov recently showed the existence of both stable and unstable periodic solutions for such a model. Their result naturally raises the question of whether it is possible to change the stability properties of such solutions.

We use the well-known Pyragas control to change the stability of periodic solutions of the thermal control model. Using this method, one adds an additional delay term to the boundary without destroying the known periodic solution. This results in a parabolic equation with both hysteresis and delay terms on the boundary. Using the Fourier decomposition, we reduce this equation to a system of ODEs. Then we can apply our finite dimensional reduction technique and show that Pyragas control can change stability of periodic solutions.

Transition Fronts for the Fisher–KPP Equation

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We are concerned with transition fronts for reaction-diffusion equations of the Fisher–KPP type. Basic examples are the standard traveling fronts, but the class of transition fronts is much larger. We will describe it and study its qualitative dynamical properties. In particular, we characterize the set of admissible asymptotic past and future speeds, as well as the asymptotic profiles, and we show that transition fronts can only accelerate. We also classify the transition fronts in the class of measurable superpositions of standard traveling fronts.

This is a joint work with François Hamel.

Continuous Dependence of Functional Differential Equations on the Scaling Parameter

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Let Ω be a bounded domain in $\mathbb{R}^n$, containing the origin. Consider the Dirichlét problem

$$-\sum_{i,j=1}^{n} \left(a_{ij}u_{x_i}(x) + b_{ij}u_{x_i}(x/p) + c_{ij}u_{x_i}(x/q)\right)_{x_j} = f(x) \quad (x \in \Omega \subset \mathbb{R}^n), \quad u|_{\partial\Omega} = 0,$$

with $a_{ij}, b_{ij}, c_{ij} \in \mathbb{C}$ $(i,j = 1,\dots,n)$ and $f \in L_2(\Omega)$. Assuming that q is a fixed number greater than 1 while p ranges over a certain segment $[1, p_0]$, we are interested in the unique solvability of the problem in the Sobolev space $\mathring{H}^1(\Omega)$ (by a solution, we mean a generalized solution understood in the standard way) for all the considered values of p and the behaviour of the family of solutions $u = u_p$ as $p \to 1$.

Exact solvability conditions in terms of the coefficients were earlier found [1] in the case where different scaling parameters were all integer powers, positive or negative, of one parameter. Now, there is no tie-up between the two parameters p and q. On the other hand, the dependence of a solution to a functional differential equation on a parameter defining the argument transformation, is studied for the first time.

Introduce the symbols (the summation is over $i, j = 1, \dots, n$)

$$a(\xi) = \sum a_{ij}\xi_i\xi_j, \quad b(\xi) = \sum b_{ij}\xi_i\xi_j, \quad c(\xi) = \sum c_{ij}\xi_i\xi_j \quad (\xi \in \mathbb{R}^n).$$

Theorem 1. *If*

$$|b(\xi)| + q^{n/2}|c(\xi)| < \operatorname{Re} a(\xi) \quad (\xi \neq 0), \tag{1}$$

then there exists a number $p_0 > 1$ such that the Dirichlét problem has a unique solution $u = u_p \in \mathring{H}^1(\Omega)$ for all values of $p \in [1, p_0]$ with $u_p \to u_1$ in $\mathring{H}^1(\Omega)$ as $p \to 1$.

The proof is based on the use of the Gårding-type inequality

$$\operatorname{Re}\sum_{i,j=1}^{n}\int_{\Omega}\left(a_{ij}u_{x_i}(x)+b_{ij}u_{x_i}(x/p)+c_{ij}u_{x_i}(x/q)\right)\bar{u}_{x_j}\,dx \geqslant \gamma\|u\|^2_{H^1(\Omega)} \quad (\forall u\in \mathring{H}^1(\Omega)).$$

Condition (1) is actually necessary and sufficient for this inequality to hold with a common constant $\gamma > 0$ for all $p \in [1, p_0]$ provided $p_0 > 1$ is sufficiently close to 1.

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Uptake and Transport of Nanoparticles and Drugs in Biological Matter

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Recent results on the uptake of inorganic nanoparticles into cells and skin are reviewed. These nanoparticles can be prepared in narrow size distributions by colloidal chemistry approaches. They were functionalized by organic ligands so that also their electrostatic stabilization and aggregation in biological media can be controlled [1]. Uptake processes into cells involve the transfer trough the cell membrane, which is commonly called endocytosis. Subsequent endocytic pathways may transfer them into different cellular compartments, such as early and late endosomes or lysosomes. In some cases the uptake into the nucleus is observed, which depends on the size of the nanoparticles and their surface properties. Experimental evidence for such transport and uptake processes comes from interdisciplinary work in combination with modern approaches in microscopy and spectromicroscopy, such as electron microscopy, X-rays microscopy and tomography, as well as fluorescence microscopy.

The uptake of variable size nanoparticles and drugs into skin is reported, as well. Various uptake routes and transport through the uppermost horny layer (stratum corneum), epidermis, and dermis can be considered, such as intercellular and transcellular transport. In addition, the role of hair follicles for the uptake of nanoparticles will be discussed [2]. Results from experimental approaches, such as confocal laser scanning microscopy and electron microscopy, as well as X-ray microscopy are presented. X-ray microscopy has the specific advantage that it combines chemical sensitivity with high spatial resolution, so that detailed information on the uptake processes of nanoparticles into cells and skin is derived [3]. Furthermore, the importance of the stratum corneum providing a strong barrier against nanoparticle uptake into skin is discussed in detail. Damage of this barrier has been induced by tape stripping, oxazolone-induced allergic contact dermatitis, and mechanical impact (pricking). The uptake of nanoparticles into damaged skin is reported, indicating that only pricked skin shows evidence for nanoparticle uptake into deeper skin layers. Detailed studies on the uptake of antiinflammatory drugs into human skin have been performed, in which the formulation of the drugs has been varied, reaching from neat drugs, gels, to nanocarriers facilitating the drug uptake into skin. Perspectives for the use of the reported results regarding novel concepts of drug delivery in topical therapy of inflammatory skin diseases will be briefly discussed.

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On Averaging of Random Semigroups and Their Generators by Using the Chernoff Theorem

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The object of investigation is the random semigroups, which are random variables whose values are one-parameter semigroups of maps of some Banach space. The mean value of such random variable is the operator-valued function on the real semiaxis which may not possess the semigroup property. The equivalence relation on the set of operator-valued functions is introduced (similarly to [1]). The equivalence of the mean value of random semigroups to some semigroup is established. The generator of the last semigroup is defined as the mean value of the random generator. This averaging procedure is a generalization of the procedure of averaging in the Banach space of bounded linear operators. The examples of application of the averaging procedure to random unbounded self-adjoint operators is considered (see [2, 3]). The parametrization of the set of limit points of a sequence of semigroups by the set of finite additive measures on the set of semigroups is given. Application of the random semigroup approach introducing the averaging procedure into the problem of blow up phenomenon in differential equations is discussed (see [4]).

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Properties of the Operators “Curl” and “Gradient of Divergence” in Sobolev Spaces

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We consider the operators “curl” and “gradient of divergence” over an arbitrary bounded domain G with smooth boundary Γ and study their properties in Sobolev spaces.

We proved the existence in the space $\mathbf{L}_2(G)$ of the orthogonal subspaces

$$\mathbf{V}^0(G) = \{\mathbf{u} \in \mathbf{L}_2(G) : \mathrm{div}\mathbf{u} = 0, \mathbf{n} \cdot \mathbf{u}|_\Gamma = 0\}$$

and

$$\mathcal{A}_\gamma(G) = \{\nabla h, h \in H^1(G) : \mathbf{n} \cdot \nabla h|_\Gamma = 0\},$$

in which the operators considered have self-adjoint realizations. Hence, each of these operators has a complete orthogonal system of eigenfunctions with nonzero eigenvalues, the **curl** in $\mathbf{V}^0(G)$ and the ∇**div** in $\mathcal{A}_\gamma(G)$, and the cumulative system is complete in $\mathbf{L}_2(G)$.

If G is a ball B, then it was proved in [5] that both eigenfunctions of the **curl** and of the ∇**div** operators could be constructed explicitly.

The necessary and sufficient conditions on $\mathbf{u}$ from $\mathbf{V}^0(B)$ and $\mathbf{v}$ from $\mathcal{A}_\gamma(B)$ are found under which their Fourier series converge in the norm of the Sobolev space $\mathbf{H}^s(B)$, $s > 0$. Namely, $\mathbf{u} \in \mathbf{V}^s_{\mathcal{R}}(B)$ and $\mathbf{v} \in \mathbf{A}^s_{\mathcal{K}}(B)$, where $\mathbf{V}^s_{\mathcal{R}}(B) = \{\mathbf{u} \in \mathbf{H}^s(B) : \mathrm{div}\mathbf{u} = 0, \mathbf{n} \cdot \mathbf{u}|_S = 0, ..., \mathbf{n} \cdot \mathrm{curl}^{s-1}\mathbf{u}|_S = 0\}$.

For the Laplace operator with the Dirichlét (Neumann) boundary condition, the spaces $H^s_{\mathcal{D}}(G)$ $(H^s_{\mathcal{N}}(G))$ was described in [2].

The problem $\mathbf{curl}\,\mathbf{u} + \lambda\mathbf{u} = \mathbf{f}$ in the ball B, $\mathbf{n} \cdot \mathbf{u}|_S = 0$, can be solved explicitly by the Fourier method [5] for any λ. For an arbitrary bounded domain G, we use the functional-analytic method [6, 7] in studying the following boundary value problems: $\mathbf{curl}\,\mathbf{u} + \lambda\mathbf{u} = \mathbf{f}$ in G, $\mathbf{n} \cdot \mathbf{u}|_\Gamma = g$, and $\nabla\mathbf{div}\,\mathbf{v} + \lambda\mathbf{v} = \mathbf{f}$ in G, $\mathbf{n} \cdot \mathbf{v}|_\Gamma = g$. For $\lambda \neq 0$, each of these problems is solvable by the Fredholm theory in appropriate Sobolev spaces.

Remark. The **curl** and the ∇**div** operators annihilate each other. Hence, $\mathbf{curl}\,\mathbf{u} + \lambda\mathbf{u} = \lambda\mathbf{u}$ if $\mathbf{u} \in \mathcal{A}_\gamma(G)$ and $\nabla\mathbf{div}\,\mathbf{v} + \lambda\mathbf{v} = \lambda\mathbf{v}$ if $\mathbf{v} \in \mathbf{V}^0(G)$. Each eigenfunction of the ∇**div** operator with eigenvalue $\mu \neq 0$ is an eigenfunction of the **curl** operator for the eigenvalue $\lambda = 0$ and vice versa.

The results have applications in hydrodynamics [1, 3–5].

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Differential Equations on Complex Manifolds

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In this talk we deal with the theory of differential equations on complex manifolds. Namely, we study linear differential equations in $\mathbb{C}^N$ and, more generally, on complex manifolds. This theory is remarkable and interesting. It turns out that in order to study equations on complex manifolds, one has to engage essentially new methods other than in the classical real analysis. These methods will be described in the talk, in particular, we shall describe the Sternin–Shatalov transform, which enables one to solve equations with constant coefficients.

The complex theory has important applications in mathematics and physics. In particular, we shall show how the methods described in the talk enable one to solve balayage inwards problem (Poincaré).

Maximal regularity of parabolic problems with operator satisfying the Kato conjecture

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Let $V \subset H \subset V'$ be complex separable Hilbert spaces with continuous and dense embeddings. We suppose that spaces V and V' are dual with respect to the inner product in H. Let $a[u,v]$ be a bounded sesquilinear form on H with domain V and let it be strongly coercitive, i.e., there exist a constant $C > 0$ such that $\operatorname{Re} a[u,u] \geqslant C\|u\|_V^2$. This form defines a bounded linear operator $A\colon V \to V'$ and unbounded linear operator $\mathcal{A}\colon H \supset D(\mathcal{A}) \to H$. A vector $u \in V$ belongs to $D(\mathcal{A})$ if there exists an element $\mathcal{A}u \in H$ such that $(\mathcal{A}u, v)_H = (Au, v)_H$ for all $v \in V$. It is known that operators A and $\mathcal{A}$ are generators of strongly continuous analytic semigroups and $Au = \mathcal{A}u$ for $u \in D(\mathcal{A})$.

If $v \in [V', V]_{1-\theta}$ for $0 \leqslant \theta \leqslant 1/2$, then the equality

$$(Au, v)_H = (A_\theta u, v)_H$$

defines a bounded linear operator $A_\theta\colon D(A_\theta) \to [V', V]'_{1-\theta} = [V', V]_\theta$ and $A_\theta u = Au$ for $u \in D(A_\theta)$ (since $[V', V]_{1-\theta}$ is dense in H).

We consider the following problem:

$$u' + A_\theta u = f, \tag{1}$$

$$u|_{t=0} = \varphi. \tag{2}$$

A function $u \in L_p(0,T; D(A_\theta)) \cap W^{1,p}(0,T;[V',V]_\theta)$ is called a solution to problem (1)-(2) if it satisfies equality (1) for almost all $t \in (0,T)$ and initial condition (2).

Condition 1. *The operator $\mathcal{A}$ satisfies the Kato conjecture, i.e. $D(\mathcal{A}^{1/2}) = V$.*

Theorem 1. *Assume that Condition 1 is satisfied, $\theta \in (0,1/2]$ and $1 < p \leqslant 1/\theta$ or $\theta = 0$ and $1 < p < \infty$, and $f \in L_p(0,T;[V',V]_\theta)$. Then problem (1)-(2) has a unique solution iff $\varphi \in (V',V)_{1-\frac{1}{p}+\theta,\,p}$.*

The work is partially supported by RFBR grants 13-01-00923 and 14-01-00265, and the President grant 4479.2014.1 for government support of the leading scientific schools of the Russian Federation.

On Homogenization for Periodic Elliptic Second-Order Differential Operators in a Strip

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We consider a homogenization problem for the differential operator

$$\mathcal{A}^\varepsilon = -\operatorname{div} g(x_1/\varepsilon, x_2)\operatorname{grad}$$

in $L_2(\mathbb{R} \times (0,a))$ with periodic boundary conditions. The matrix-valued function g is supposed to be bounded and positive definite, periodic in the first variable and Lipschitz continuous in the second one. The aim is to study the limit behavior of $\mathcal{A}^\varepsilon$ as $\varepsilon \downarrow 0$. We prove that the resolvent of $\mathcal{A}^\varepsilon$ converges in the operator norm to the resolvent of an operator of a similar form with coefficients independent of x_1, and obtain a sharp bound for the difference.

Harmonic Spheres and Yang–Mills Fields

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We consider a relation between harmonic spheres in loop spaces and Yang–Mills fields on the Euclidean 4-space.

Harmonic spheres are given by smooth mappings of the Riemann sphere into Riemannian manifolds, being the extremals of the energy functional given by the Dirichlét integral. They satisfy nonlinear elliptic equations generalizing the Laplace–Beltrami equation. If the target Riemannian manifold is a Kähler one, then holomorphic and anti-holomorphic spheres realize local minima of the energy. On the other hand, Yang–Mills fields are the extremals of theYang–Mills action functional. Local minima of this functional are called instantons and anti-instantons.

Atiyah proved that the moduli space of G-instantons on $\mathbb{R}^4$ can be identified with the space of based holomorphic spheres in the loop space ΩG of a compact Lie group G. In our talk, we discuss a conjecture generalizing Atiyah's theorem, asserting that there must be a bijective correspondence between the moduli space of Yang–Mills G-fields on $\mathbb{R}^4$ and the space of based harmonic spheres in the loop space ΩG.

Asymptotic Solution of Linear System of Differential Equations for the Water Waves in a Basin With Fast Oscillating Bottom

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We consider a linear system of differential equations modelling water waves in a basin with the fast oscillating bottom. The bottom has the form of a slow changing background with the additional fast oscillation. Using the homogenization method in an operator form, we reduce the original problem to a pseudodifferential equation for the potential on the free surface [1–3]. This equation includes two types of dispersion. The first one is the standard water wave dispersion and the second one is the anomalous dispersion connected with the fast oscillation of the bottom. We compare the influence of these two dispersions on the different types of waves.

This work was done together with S. Yu. Dobrokhotov and B. Tirozzi and was supported by grant of the President of the Russian Federation No. MK-1017.2013.1, RFBR grant No. 14-01-00521-a, and project RIMARE (CINFAI-Italy).

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The Classical KAM Theory in the Last Decade: a Slow Progress

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In the talk, I plan to present a survey of some important results of the last decade (2005–2014) in the finite dimensional Kolmogorov–Arnold–Moser theory. These results indicate a progress slow compared with the "heroic" period of the sixties but definitely not with the preceding decade of 1995–2004.

The main achievement in KAM theory of the last decade is of course the so-called *"a posteriori"* approach, where an approximately invariant torus implies the existence of a genuinely invariant torus nearby while the system is not assumed to be nearly integrable in any sense. This powerful setup also known as "KAM theory without action-angle variables" (one also speaks of "parameterization methods") has been developed for quasi-periodic motions in many various contexts (R. de la Llave, A. González-Enríquez, À. Jorba, J. Villanueva, E. Fontich, Y. Sire, G. Huguet, A. Luque, À. Haro, H. N. Alishah, R. C. Calleja, A. Celletti, etc.)

Three particular recent examples of KAM-like theorems in an "a posteriori" format are of special interest. The first one is the results on KAM tori in *presymplectic* dynamical systems (a presymplectic structure is a closed 2-form of constant rank) due to H. N. Alishah and R. de la Llave (2012). The second example is KAM theory for *conformally symplectic* systems (systems that transform the symplectic form into its multiple) due to R. C. Calleja, A. Celletti, and R. de la Llave (2013). The third one is the so-called singularity theory for non-twist KAM tori (tori with degenerate torsion) due to A. González-Enríquez, À. Haro, and R. de la Llave (2014), an impressive combination of KAM theory and the bifurcation theory for critical points of functions.

Besides the "a posteriori" methodology, I plan to consider some isolated topics of more conventional nature. One of them concerns counterexamples of smoothness classes $C^{2n-\varepsilon}$ to the KAM theorem for nearly integrable Hamiltonian systems with n degrees of freedom (Ch.-Q. Cheng and L. Wang, 2011–2013). Another topic is hyperbolic lower dimensional invariant tori in Hamiltonian systems with proper degeneracy (A. G. Medvedev, 2013). One more example is KAM theory for the so-called *reversible context 2*, where the dimension of the fixed point manifold of the reversing involution is less than half the codimension of the invariant torus in question (M. B. Sevryuk, 2011–2012).

Asymptotic Behavior of Critical Points of Energy Involving "Circular-Well" Potential

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We study the singular limit of critical points of an energy with a penalization term depending on a small parameter. The energy involves a potential which is a nonnegative function on the plane, vanishing on a closed curve. We generalize to this setting the results obtained by Bethuel, Brezis, and Helein for the Ginzburg–Landau energy.

This is a joint work with Petru Mironescu (Lyon I).

Application of the Method of Characteristics to Construct Generalized Solutions of the Hamilton–Jacobi Equation with State Constraints

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The following Cauchy problem with state constraints for the Hamilton–Jacobi equation is considered:

$$\partial u/\partial t + H(x, \partial u/\partial x) = 0, \quad t \geqslant 0, \quad x \in [-1; 1]) \tag{1}$$

$$H(x,p) = -f(x) + 1 - \frac{1+x}{2} e^{2p} - \frac{1-x}{2} e^{-2p}, \tag{2}$$

$$u(0,x) = u_0(x), \quad x \in [-1; 1]. \tag{3}$$

The problem arises in the Crow–Kimura model of evolution genetics [1]. A concept of continuous generalized solutions to the problem with state constraints is suggested [2]. The solutions are introduced with the help of viscosity and minimax solutions to auxiliary Dirichlét problems. Construction of generalized solutions is based on optimal control theory and dynamic programming. This approach can be considered as a generalization of the classical Cauchy method of characteristics.

The characteristic system of problem (1)-(3) is considered,

$$\begin{aligned} \dot{x} = H_p(x,p) &= -(1+x)e^{2p} + (1-x)e^{-2p}, \\ \dot{p} = -H_x(x,p) &= f'(x) + (e^{2p} - e^{-2p})/2, \\ \dot{z} &= pH_p(x,p) - H(x,p), \end{aligned} \tag{4}$$

where $H_x(x,p) = \partial H(x,p)/\partial x$ and $H_p(x,p) = \partial H(x,p)/\partial p$.

Properties of the characteristics are studied. They are then used to construct generalized solutions. Simulation results are given.

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Two-Point Boundary Value Problem for Systems of Impulsive Differential Equations of Fractional Order

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Differential equations of fractional order have proved to be a valuable tool in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, dynamical processes in self-similar structures, biosciences, signal processing, control theory, electrochemistry, and diffusion processes (see [1] for references therein).

Impulsive differential equations of fractional order play an important role in theory and applications. Many evolutionary processes are characterized by abrupt changes of states at certain time instants.

In this work, we consider existence and uniqueness of nonlinear fractional impulsive differential equations of the type

$$ {}^{c}D_{0+}^{\alpha}x\left(t\right)=f\left(t,x\left(t\right)\right)\ \ for\ a.e.\ t\in\left[0,T\right],\ \ t\neq t_i, i=1,2,...,p, \tag{1}$$

subject to the two-point boundary conditions

$$A_1 x\left(0\right)+B_1 x\left(T\right)=C_1, \tag{2}$$

$$A_2 x'\left(0\right)+B_2 x'\left(T\right)=C_2 \tag{3}$$

and the impulsive conditions

$$\begin{array}{l}\Delta x\left(t_i\right)=I_i(x(t_i)),\quad \Delta x'\left(t_i\right)=I_i^*(x(t_i))\, i=1,2,...,p,\\ 0=t_0<t_1<t_2<...<t_p<t_{p+1}=T,\end{array} \tag{4}$$

where A_k and $B_k\in\mathbb{R}^{n\times n}$ are given matrices with $\det\left(A_k+B_k\right)\neq 0$, $k=1,2$, and $C_1,C_2\in\mathbb{R}^n$.

Here $f:[0,T]\times\mathbb{R}^n\to\mathbb{R}^n$, I_i, $I_i:\mathbb{R}^n\to\mathbb{R}^n$, and I_i^*, $I_i^*:\mathbb{R}^n\to\mathbb{R}^n$, $i=1,2,...,p$, are given functions, ${}^{c}D_{0+}^{\alpha}$ is the Caputo fractional derivative of order α, $0<\alpha\leqslant 2$, $\Delta x(t_i)=x(t_i^+)-x\left(t_i^-\right)$, $x\left(t_i^+\right)=\lim\limits_{h\to 0+}x\left(t_i+h\right)$, and $x\left(t_i^-\right)=\lim\limits_{h\to 0+}x\left(t_i-h\right)=x\left(t_i\right)$ are the right-hand and the left-hand limits of $x\left(t\right)$ at $t=t_i$, respectively.

In this work, the sufficient conditions are established for the existence and uniqueness of a solution to the boundary value problem. Note that problem (1)–(4) generalizes the problem considered in [2]. However, the results obtained in [2] contain serious mistakes.

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Very Singular and Large Solutions of Semi-Linear Parabolic and Elliptic Equations with Degenerate Absorption Potential

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As a model problem, we study existence and limit behavior as $k \to \infty$ of solutions u_k of the following Cauchy problem: $u_t - \Delta u + hu^\lambda = 0$ in $\Omega = \mathbb{R}^N \times R^1_+$, $u(x,0) = k\delta_0(x)$, $\lambda > 1$, $N \geqslant 1$, where $\delta_0(x)$ is the Dirac measure and the function $h = h(x,t) \geqslant 0$ vanishes on some smooth manifold Γ with $(0,0) \in \Gamma$. So, if $h = h(|x|) = |x|^\beta$, $\beta > 0$ ($\Gamma = \{0,t\}$), and $1 < \lambda < \lambda_{\text{cr}} := 1 + N^{-1}(2+\beta)$, then for any $k \in \mathbb{N}$ there exists the "fundamental" solution u_k, and the limit function u_∞ is obviously a very singular (more singular than u_k) solution with a point singularity at $(0,0)$. If $\lambda \geqslant \lambda_{\text{cr}}$, then the problem considered has no solution for any k.

Stronger degeneration of the potential h yields new phenomenon. As $k \to \infty$, the point singularity of the solutions u_k may spread to the whole manifold Γ and, as a result, u_∞ becomes a solution only in $\Omega \backslash \Gamma$ with the singularity set Γ (the "razor blade" solution being an example). For some model manifolds Γ, in particular, $\Gamma_1 = \{0,t\}$ or $\Gamma_2 = \{x,0\}$, we found a criterion on the flatness of h near Γ, guaranteeing the above mentioned propagation.

We investigate this phenomenon for different classes of quasilinear parabolic and elliptic equations of diffusion-degenerate strong absorption type. Moreover, the above theory of very singular solutions is generalized to the case of so-called "large" solutions of corresponding semilinear elliptic and parabolic equations too.

Some results are obtained jointly with Laurent Veron or Moshe Marcus.

Numerical Range of Holomorphic Mappings in Geometric Function Theory

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The study of the numerical range of a holomorphic mapping arises in many aspects of nonlinear analysis, finite and infinite dimensional holomorphy, and complex dynamical systems. In particular, such notion plays a crucial role in establishing exponential and product formulas for semigroups of holomorphic mappings, the study of flow invariance and range conditions, geometric function theory in finite and infinite dimensional Banach spaces, and in the study of complete and semi-complete vector fields and their applications to starlike and spiral-like mappings and the Bloch (univalence) radii for locally biholomorphic mappings.

In addition, we discuss some geometrical and quantitative analytic aspects of the fixed point theory. We present a solution of the so-called coefficient problem in branching stochastic processes by using the growth estimates for the numerical range of holomorphic mappings.

A New Notion of Entropy Which is Local in Time and Space

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The ideas of metric entropy go back to Shannon's information theory and describe a measure of information loss for ergodic transformations. In many applied systems, ergodicity can not be assumed. In fact, often the notion of ergodicity itself is meaningless since the dynamics is time-varying and known only on a bounded time interval, e.g., if satellite data of the velocity field of the ocean surface is available or a forecast of a fluid flow for the next couple of minutes is computed. In this talk, we introduce a new concept of finite-time entropy which is a local version of the classical concept of metric entropy. Based on that, a finite-time version of Pesin's entropy formula is derived. We will also talk about how to apply the finite-time entropy field to detect special dynamical behavior such as Lagrangian coherent structures.

This is a joint work with Luu Hoang Duc.

Nonlocal Elliptic Problems and Applications to Vlasov–Poisson Equations

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An interest to the Vlasov equations is connected with important applications. In particular, these equations describe dynamical processes in controlled thermonuclear reactor. During last years, a great progress was achieved in investigation of the Vlasov equations, see A. A. Arsen'ev, J. Batt, Y. Guo, E. Horst, P. L. Lions, V. P. Maslov, K. Pfaffelmoser, J. Schäffer, C. Villani, V. V. Vedenyapin, etc. For bibliography, see [1]. In order to create stable plasma of high temperature in a reactor, it is necessary to hold the plasma fuse strictly inside the domain. The majority of models of thermonuclear reactors make use of control in the form of an external magnetic field to provide the existence of plasma in the reactor. In terms of differential equations, this means that we provide the existence of compactly supported solutions to the Vlasov–Poisson equations with respect to distribution functions of densities for charged particles, using an external magnetic field. However, the influence of an external magnetic field on the trajectories of the particles was not considered in mathematical papers.

We consider the Vlasov–Poisson system of equations with an external magnetic field, describing the evolution of distribution functions of densities for charged particles in a rarefied plasma. We study the Vlasov–Poisson system in $Q \times \mathbb{R}^3$ with the initial conditions $f^\beta\big|_{t=0} = f_0^\beta(x, v)$, $\beta = \pm 1$, for the distribution functions $f^\beta(x, v, t)$ and the nonlocal boundary conditions for the potential of an electric field $\varphi(x, t)$ where $Q = \Omega \times \mathbb{R}$, $\Omega \subset \mathbb{R}^2$ is a bounded domain, $\partial\Omega \in C^\infty$, $f_0^\beta(x, v)$ is the initial distribution function (for positively charged ions if $\beta = +1$ and for electrons if $\beta = -1$) at a point x with velocity v. Assume that the initial distribution functions are sufficiently smooth and $\operatorname{supp} f_0^\beta \subset Q_\delta \times B_\rho(0)$, $Q_\delta = \{x \in Q \colon \rho(x, \partial Q) > \delta\} \neq \varnothing$, $\delta, \rho > 0$, and the magnetic field $B(x)$ has the form $(0,0,h)$, where $h > 0$ is sufficiently large. We construct a stationary solution of the Vlasov–Poisson system $\{0, \mathring{f}^{\beta}(x,v)\}$ such that $\mathrm{supp}\mathring{f}^{\beta}(x,v) \subset Q_{\delta} \times B_{\rho}(0)$. Then we prove that for any $T > 0$ there is a unique classical solution of the Vlasov–Poisson system in $Q \times \mathbb{R}^3$ for $0 < t < T$ if $\|f_0^{\beta} - \mathring{f}^{\beta}\| < \varepsilon$, where $\varepsilon = \varepsilon(T, \delta, \rho, h)$ is sufficiently small. For the proof, we establish a theorem on existence and uniqueness of solutions to nonlocal elliptic problem in an infinite cylinder in Höleder spaces and use the characteristics method and the shapes of the Larmor trajectories in homogeneous magnetic field.

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Simulation of Dynamic Systems Describing the Rogue Waves

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In our talk, we consider dynamical systems describing water waves. It is well known that dynamics of waves on water leads to formation of rogue waves. We consider rogue waves area in the phase space. It is proved that probability of emergence of a rogue wave is equal to the measure of rogue waves area in the phase space. The measure of the set respective to rogue waves is estimated by means of computing experiments.

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On Pseudomonotone Property for Differential-Difference Operators in Elliptic Equations of Variational Type

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Let $Q \subset \mathbb{R}^n$ be a bounded domain with smooth boundary or $Q = (0,d) \times G$, where $G \subset \mathbb{R}^{n-1}$ is a bounded domain (with smooth boundary if $n \geqslant 3$).

Let us consider the problem

$$R^*\mathcal{A}Ru(x) = f(x) \quad (x \in Q) \tag{1}$$

with the boundary condition

$$u(x) = 0 \quad (x \notin Q). \tag{2}$$

Here

$$\mathcal{A}u(x) = -\sum_{1\leqslant i\leqslant n} \partial_i A_i\left(x, u, \partial_1 u, \cdots, \partial_n u\right),$$

A_i are sufficiently smooth functions such that the operator $\mathcal{A}$ is pseudomonotone, demicontinuous, and coercive,

$$Ru(x) = \sum_{h\in\mathcal{M}} a_h u(x+h),$$

$\mathcal{M}$ is a finite set of vectors with integer coordinates, and $a_h \in \mathbb{R}^n$ (the case of commensurable shifts is treated similarly).

As is well known, an elliptic operator equation has a solution if the corresponding operator is pseudomonotone, demicontinuous, and coercive. We prove that if R is nondegenerate in some sense and $\mathcal{A}$ has the above mentioned properties then the operator $R^*\mathcal{A}R$ has the same properties. Thus, problem (1)-(2) has a solution.

Note that linear elliptic differential–difference equations were studied in [1].

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Embedding Inequality and Duality Principle in Weighted Sobolev Spaces on the Semiaxis

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We analyze a characterization of an embedding inequality of Sobolev type with the help of the duality principle and boundedness criteria for the Hardy–Steklov integral operator in weighted Lebesgue spaces.

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Extinction of Solutions for $2m$-Order Nonlinear Parabolic Equations

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The talk is devoted to the solutions' behavior for the problem

$$\left(|u|^{q-1}u\right)_t + (-1)^m \sum_{|\eta|=m} D_x^\eta \left(|D_x^m u|^{q-1} D_x^\eta u \right) + a(x)|u|^{\lambda-1}u = 0 \quad \text{in} \quad Q, \qquad (1)$$

$$D_x^\eta u \Big|_{(0,+\infty)\times\partial\Omega} = 0 \qquad \forall \eta : |\eta| \leqslant m-1, \qquad (2)$$

$$u(0,x) = u_0(x), \qquad x \in \Omega, \qquad (3)$$

where $Q = (0,+\infty)\times\Omega$, $\Omega \subseteq \mathbb{R}^N, N \geqslant 1$, $m \geqslant 1$, and $0 \leqslant \lambda < q$. We assume that the absorption potential $a(x)$ is a nonnegative measurable function bounded in Ω. The main assumptions on the degeneration of $a(x)$ are

$$\int_0^c \frac{(meas\{x \in \Omega : \ a(x) \leqslant s\})^\theta}{s} ds < +\infty \qquad \forall\, c > 0, \qquad (4)$$

where $\theta = \min\left(\frac{m(q+1)}{N}, 1\right)$, $N \neq 2m$, and

$$\int_0^c \frac{meas\{x \in \Omega : \ a(x) \leqslant s\}(-ln\, meas\{x \in \Omega : \ a(x) \leqslant s\})}{s} ds < +\infty \quad \forall\, c > 0 \qquad (5)$$

for $N = m(q+1)$.

If $a(x)$ satisfies (4) or (5), then the set where this function takes small values is small enough. For instance, if $a(x) \geqslant \gamma = const > 0$, then the integrals are finite. On the contrary, if $a(x) = 0$ on a set of positive measure, the integrals become infinite. The main results is the following.

Theorem.

a) *If* $N \neq 2m$ *and* (4) *holds, then any solution of problem* (1)–(3) *has the extinction in finite time.*

b) *If* $N = 2m$ *and* (5) *holds, then any solutions of problem* (1)–(3) *has the extinction in finite time.*

Attraction Property of Local Center-Unstable Manifolds for Differential Equations with State-Dependent Delay

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We consider a class of functional differential equations of the form

$$\dot{x}(t) = f(x_t) \tag{1}$$

with f defined on an open subset of $C^1([-h, 0], \mathbb{R}^n)$, $h > 0$. Under certain conditions, which are typically satisfied in cases where Eq. (1) represents an autonomous differential equation with state-dependent delay, the associated Cauchy problems define a smooth semiflow on a submanifold of $C^1([-h, 0], \mathbb{R}^n)$. In particular, it is known that at a stationary point of the semiflow there exist so-called local center-unstable manifolds. Here we discuss the attraction property of these manifolds. More precisely, we prove that for a fixed local center-unstable manifold W_{cu} of Eq. (1) at a stationary point ϕ, each solution that exists and remains close enough to ϕ for all $t \geqslant 0$ converges exponentially as $t \to \infty$ to a solution on the local center-unstable manifold W_{cu}.

Random Perturbations in the Problem of Capture into Autoresonance

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The object of our research is the system of two differential equations

$$\frac{dr}{d\tau} = \sin\psi, \quad r\Big[\frac{d\psi}{d\tau} - r^2 + \tau\Big] = \cos\psi.$$

These equations describe the initial stage of capture into autoresonance for different nonlinear oscillating systems [1]. Here $r(\tau)$ and $\psi(\tau)$ are slow varying amplitude and phase shift of fast harmonic oscillations. Solutions whose amplitude increase in time $r(\tau) \approx \sqrt{\tau}$ correspond to autoresonance.

The perturbed equations are considered in the form

$$\frac{dr}{d\tau} = \sin\psi + \mu\xi, \quad r\Big[\frac{d\psi}{d\tau} - r^2 + \tau\Big] = \cos\psi + \mu\eta, \quad 0 < \mu \ll 1.$$

Here $\xi(r, \psi, \tau; \omega)$ and $\eta(r, \psi, \tau; \omega)$ correspond to random perturbations or noise. The problem is to describe the class of perturbations under which the capture into autoresonance occurs [2].

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Operator Error Estimates for Homogenization of Elliptic Systems with Periodic Coefficients

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Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain of class $C^{1,1}$. In $L_2(\mathcal{O};\mathbb{C}^n)$, we consider matrix elliptic second order differential operators (DO's) $A_{\flat,\varepsilon}$, $\flat = D, N$, given by $b(\mathbf{D})^* g(\mathbf{x}/\varepsilon) b(\mathbf{D})$, with the Dirichlét or Neumann boundary conditions, respectively. Here $0 < \varepsilon \leqslant 1$ and $g(\mathbf{x})$ is an $(m \times m)$-matrix-valued function that is bounded, uniformly positive definite and periodic with respect to some lattice Γ. Next, $b(\mathbf{D})$ is a first order $(m \times n)$-matrix DO with constant coefficients, where $m \geqslant n$. The symbol $b(\boldsymbol{\xi})$ is subject to some condition which ensuring that $A_{\flat,\varepsilon}$ is strongly elliptic.

We study the behavior of the resolvent $(A_{\flat,\varepsilon} - \zeta I)^{-1}$ for small ε, where ζ is a regular point.

Theorem (see [1–3]). *There exists a number $\varepsilon_0 \in (0,1]$ depending on the domain $\mathcal{O}$ and the lattice Γ such that for $0 < \varepsilon \leqslant \varepsilon_0$ we have*

$$\|(A_{\flat,\varepsilon} - \zeta I)^{-1} - (A_\flat^0 - \zeta I)^{-1}\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \leqslant C_{1,\flat}(\zeta)\varepsilon, \tag{1}$$

$$\|(A_{\flat,\varepsilon} - \zeta I)^{-1} - (A_\flat^0 - \zeta I)^{-1} - \varepsilon K_\flat(\varepsilon;\zeta)\|_{L_2(\mathcal{O}) \to H^1(\mathcal{O})} \leqslant C_{2,\flat}(\zeta)\varepsilon^{1/2},$$

$\flat = D, N$. *Here $A_\flat^0$, $\flat = D, N$, is the effective operator given by $b(\mathbf{D})^* g^0 b(\mathbf{D})$ with the Dirichlét or Neumann boundary conditions respectively, g^0 is the effective matrix, and $K_\flat(\varepsilon;\zeta)$ is the corresponding corrector.*

Estimate (1) is order sharp. The positive constants $C_{1,\flat}(\zeta)$, $C_{2,\flat}(\zeta)$ are controlled in terms of the problem data.

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Coerciveness Conditions for the Functional-Differential Equations with Orthotropic Contractions

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Let Ω be a bounded domain in $\mathbb{R}^2$ containing the origin. We consider the Dirichlét problem for a functional differential equation in Ω,

$$A_R u \equiv -\sum_{i,j=1}^{2} (R_{ij} u_{x_i})_{x_j} = f(x) \quad (x \in \Omega), \qquad u|_{\partial\Omega} = 0,$$

where

$$R_{ij}v(x) = a_{ij0}v(x) + a_{ij1}v(q^{-1}x_1, px_2) + a_{ij,-1}v(qx_1, p^{-1}x_2),$$

$p, q > 1$, $a_{ij0}, a_{ij,\pm 1} \in \mathbb{C}$ $(i, j = 1, 2)$, and $f \in L_2(\Omega)$. We introduce the following notations:

$$\alpha_{i0} = \operatorname{Re} a_{ii0}\ (i = 1, 2), \quad \alpha_{11} = p^{-1}a_{111} + q^{-1}\bar{a}_{11,-1}, \quad \alpha_{21} = qa_{221} + p\bar{a}_{22,-1},$$

$$\beta_0 = \operatorname{Re}(a_{120} + a_{210}), \quad \beta_1 = qa_{121} + q^{-1}\bar{a}_{12,-1} + p^{-1}a_{211} + p\bar{a}_{21,-1}.$$

The main result of this work is contained in the theorem below.

Theorem. *The Gårding-type inequality*

$$\operatorname{Re}(A_R u, u)_{L_2(\Omega)} \geqslant c_1\|u\|^2_{W_2^1(\Omega)} - c_2\|u\|^2_{L_2(\Omega)} \tag{1}$$

holds on the set $u \in C_0^\infty(\Omega)$ *with constants* $c_1 > 0$ *and* $c_2 \geqslant 0$ *independent of* u *if and only if* $\beta_0^2 < \alpha_{10}\alpha_{20}$ *and the self-adjoint difference operators*

$$I + g^{\pm}(\tau)T + T^{-1}\bar{g}^{\pm}(\tau) : L_2(\mathbb{R}) \to L_2(\mathbb{R}), \tag{2}$$

where $Tw(\tau) = w(\tau - \ln\sqrt{pq})$ *and*

$$g^{\pm}(\tau) = \frac{\alpha_{11}e^{2\tau} + \alpha_{21}e^{-2\tau} \pm \beta_1}{2\sqrt{(\alpha_{10}e^{2\tau} + \alpha_{20}e^{-2\tau} \pm \beta_0)((pq)^{-1}\alpha_{10}e^{2\tau} + pq\alpha_{20}e^{-2\tau} \pm \beta_0)}},$$

are positive definite.

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On a Nonlinear Climate Model with Dynamic and Diffusive Boundary Condition

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We study a global climate model for the coupling of the mean surface temperature with the deep ocean temperature. The nonlinear model presents a dynamic and diffusive boundary condition representing the mean surface temperature. The model includes other nonlinear terms such as the coalbedo effect and the latent heat, which here are formulated in terms of suitable (multivalued) maximal monotone graphs. We prove the existence of bounded weak solutions via fixed point technics and show some numerical experiments. This is a joint work with J. I. Diaz (UCM, Spain) and A. Hidalgo (UPM, Spain).

Vector Fields with Shadowing Properties Corresponding to Various Classes of Reparametrisations

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The shadowing problem is related to the following question:

Under what conditions does any pseudotrajectory of a vector field have a close exact trajectory?

Reparametrisations of exact trajectories play an important role in this notion. We consider two natural classes of reparametrisations and provide an example showing that the corresponding shadowing properties are not equivalent.

An example is a non-structurally stable 4-dimensional vector field based on a special 2-dimensional vector field whose trajectories look like spirals.

Separatrix Map and Hamiltonian Dynamics

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We discuss definition and basic applications for the Zaslavsky separatrix map in Hamiltonian dynamics with 2 degrees of freedom.

On Inverse Nodal Problem for Sturm–Liouville Problem

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Let $q \in L[0,\pi]$ and $\lambda_n = \lambda_n[q]$ be n-th eigenvalue of the regular Sturm–Liouville problem

$$\begin{cases} \hat{y}'' + [\lambda - q]\hat{y} = 0, \\ \sin\alpha\hat{y}'(0) + \cos\alpha\hat{y}(0) = 0, \\ \sin\beta\hat{y}'(\pi) + \cos\beta\hat{y}(\pi) = 0, \end{cases} \tag{1}$$

where $\alpha, \beta \in \mathbb{R}$ and $\hat{y}(x,q,\lambda_n) \equiv \hat{y}_n(x)$ is the corresponding orthonormal eigenfunction of this problem. The zeroes of each function $\hat{y}_n$ are put in ascending order, $0 \leqslant x_{0,n} < x_{1,n} < \cdots < x_{n,n} \leqslant \pi$. By $x_{k,n}[q]$ we denote the functional mapping taking the potential q to $(k+1)$-th zero of n-th eigenfunction $\hat{y}(x,q,\lambda_n[q])$. We indicate by

$$D\phi[q,w] = \lim_{t\to 0} \frac{\phi(q+tw) - \phi(q)}{t}$$

the Gâteaux differential of a functional $\phi : L[0,\pi] \to \mathbb{R}$ with an increment $w \in L[0,\pi]$. We assume that the normalization condition

$$\int_0^\pi q(x)\,dx = 0 \tag{2}$$

holds true. We define a mapping $\delta[f](x)$ by assigning the real number

$$\delta[f](x) = \lim_{\varepsilon\to 0} \int_0^\pi f(\tau)\Psi(\tau,x,\varepsilon)\,d\tau$$

to each summable function f on the segment $[0,\pi]$, where $E(x,\varepsilon) = [x-\varepsilon, x+\varepsilon]\cap[0,\pi]$ and $\Psi(\tau,x,\varepsilon) = \begin{cases} \frac{1}{mesE(x,\varepsilon)}, & \text{for } \tau \in E(x,\varepsilon), \\ 0, & \text{for } \tau \in [0,\pi]\setminus E(x,\varepsilon). \end{cases}$

We denote by

$$D\phi[q,\delta[1](x)] = \lim_{\varepsilon\to 0}\lim_{t\to 0} \frac{\phi(q+t\Psi(\cdot,x,\varepsilon)) - \phi(q)}{t}$$

the action of the Gâteaux differential of a functional ϕ on an element $q \in L[0,\pi]$ with the increment $\delta[1](x)$.

Theorem 1 (see [1])**.** *Let $\mathbb{M}$ be an arbitrary set dense in $[0,\pi]$, $x_{k,n} \in (0,\pi)$ be a zero of an eigenfunction of Sturm–Liouville problem* (1)*, and the Gâteaux differential of the functional $x_{k,n}[q]$ on an element $q \in L[0,\pi]$ with the increment $\delta[1](x)$ take the value $Dx_{k,n}[q,\delta[1](x)]$ at each point x of set $\mathbb{M}$.*

Then the potential of Sturm–Liouville problem (1)*, satisfying the normalization condition* (2)*, can be represented as*

$$q(x) \overset{a.e.}{=} \frac{d^2\sqrt{\left|\lim_{x_p \overset{\mathbb{M}}{\to} x} Dx_{k,n}[q,\delta[1](x_p)]\right|}}{dx^2}\left(\sqrt{\left|\lim_{x_p \overset{\mathbb{M}}{\to} x} Dx_{k,n}[q,\delta[1](x_p)]\right|}\right)^{-1}$$

$$-\frac{1}{\pi}\int_0^\pi \left\{ \frac{d^2\sqrt{\left|\lim_{x_p \overset{\mathbb{M}}{\to} x} Dx_{k,n}[q,\delta[1](x_p)]\right|}}{dx^2}\left(\sqrt{\left|\lim_{x_p \overset{\mathbb{M}}{\to} x} Dx_{k,n}[q,\delta[1](x_p)]\right|}\right)^{-1} \right\} dx,$$

where $\{x_p\}_{p=1}^\infty$ is an arbitrary sequence converging to x along the set $\mathbb{M}$, i.e. $x_p \to x$, $x_p \in \mathbb{M}$.

The statement of Theorem 1 is non-improvable in the sense that it is impossible to neglect the density of the set $\mathbb{M}$ in $[0,\pi]$.

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Some Nonexistence Results for Anisotropic Elliptic and Backward Parabolic Inequalities

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We study the nonexistence results for some elliptic and parabolic inequalities governed by an anisotropic operator of the types

$$\sum_{i=1}^{N} \sum_{\ell_i \leqslant \alpha_i \leqslant L_i} D_{x_i}^{\alpha_i} (A_{\alpha_i}(x,u)) \geqslant f(x,u),\ x \in \mathbb{R}^N,$$

and

$$u_t + \sum_{i=1}^{N} \sum_{k_i \leqslant \beta_i \leqslant K_i} D_{x_i}^{\beta_i} (B_{\beta_i}(x,t,u)) \geqslant g(x,t,u),\ (x,t) \in \mathbb{R}^N \times \mathbb{R}_+,$$

where $\ell_i, L_i, k_i, K_i \in \mathbb{N}$ are such that $\ell_i, k_i > 1$ for each $i = 1, 2, \ldots, N$ and $A_{\alpha_i}(x,u)$, $B_{\beta_i}(x,t,u), f(x,u)$, and $g(x,t,u)$ are given Caratheodory functions.

Under appropriate conditions on the functions A_{α_i}, B_{β_i}, f, and g, we prove some nonexistence theorems for solutions to these inequalities. Our proofs are based on the nonlinear capacity method developed by E. L. Mitidieri and S. I. Pohozaev [1]. This method consists in obtaining a priori estimates based on the weak formulation of the problems with a special choice of test functions.

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On General Boundary Value Problems for Pseudodifferential Equations in a Cone

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We suggest a variant of the theory of boundary value problems for pseudodifferential equations in non-smooth domains based on a special factorization of the elliptic symbol of a pseudodifferential operator. Earlier, the author considered in details the two-dimensional situation [1,2]. It was established as a result that for well-posedness of the boundary value problem even in a canonical domain, one needs to add supplementary conditions (the Shapiro–Lopatinskii conditions at smoothness points). In the two-dimensional case, this is the solvability condition for some special system of linear difference equations [3]. In multi-dimensional cone, this is the solvability condition for a certain system of integral equations which can be obtained from operator symbols and boundary conditions.

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Quasilinear Lane–Emden Equations with Measures Data

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We study the existence of solutions to the equation

$$-\Delta_p u + g(x, u) = m,$$

where $g(x, .)$ is a nondecreasing function and m is a measure. We characterize the good measures, i.e. the ones for which the problem has a renormalized solution. We study particularly the cases where

$$g(x, u) = |x|^{-b}|u|^{q-1}u$$

and

$$g(x, u) = \operatorname{sgn}(u)(e^{t|u|^s} - 1).$$

The results state that a measure is good if it is absolutely continuous with respect to an appropriate Lorentz–Bessel capacity. Our construction is based upon sharp potential estimates.

Comparison Principles for Degenerate Elliptic Operators with Applications to Removable Singularities

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In this talk we present comparison results concerning fully nonlinear, possibly degenerate, elliptic equations, in which inequality between solutions is required only on part of the boundary, with information on the size of the remaining part. From this we deduce sufficient conditions on the size of a set in order to be a removable singularity of a solution as well as on the growth of the solution when approaching the singular set. Following Harvey and Lawson, we also consider some cases in which no assumption is needed on the growth of the solution.

Spectral Analysis of Integro-Differential Equations in Hilbert Space and Its Applications

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We study Volterra-type integro-differential equations with unbounded operator coefficients in Hilbert spaces. These equations represent the abstract forms of integro-differential equations arising in viscoelasticity theory and Gurtin–Pipkin type integro-differential equations describing the process of heat conduction in media with memory and arising in homogenization problems in perforated media.

Spectral problems for operator-functions being symbols of these equations are analyzed. The spectra of an abstract integro-differential equation is investigated. Representations for solutions of such type equations are obtained.

We generalize and extend the results obtained in [1–3].

The talk is based on the joint works with N. A. Rautian and R. P. Ortiz.

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On the Continuous Limit for Systems of Difference Equations

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Consider the system

$$Y(h, z+h) = A(z)Y(h, z), \tag{1}$$

where $z \in \mathbb{C}$, $h \in \mathbb{R}_+$, $Y(h, z) \in \mathrm{GL}(n, \mathbb{C})$, $A(z)$ is a polynomial matrix. We study the limit

$$Y(z) = \lim_{h \to 0} Y(h, z),$$

which is called a continous limit (see [2]). This system has two fundamental solutions $Y_l(h, z)$ and $Y_r(h, z)$ corresponding to one formal asymptotic solution

$$\hat{Y}(h, z) = \left(I + \frac{\hat{Y}_1}{z} + \frac{\hat{Y}_2}{z^2} + \dots \right) \mathrm{diag}(\rho_1^z z^{d_1}, \dots, \rho_n^z z^{d_n})$$

in the left ($\mathrm{Re} z \to -\infty$) and in the right ($\mathrm{Re} z \to +\infty$) half-plane, respectively (see [1]).

The system

$$\frac{dY}{dz} = B(z)Y(z), \qquad B(z) = \frac{A(z) - I}{h} \tag{2}$$

of linear differential equations is a limit (as $h \to 0$) of system (1). System (2) has solutions $Y_l(z)$ and $Y_r(z)$ provided with limit asymptotic expansions in the left and in the right half-plane, respectively.

The talk is devoted to the following theorem.

Theorem. *In the generic case, solutions $Y_l(h, z)$ and $Y_r(h, z)$ tend to solutions $Y_l(z)$ and $Y_r(z)$ of the limit system of differential equations*

$$\lim_{h \to 0} Y_{l,r}(h, z) = Y_{l,r}(z).$$

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Classification and Analysis of $SU(n+1)$ Toda System

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Using PDE method, we first give a complete classification of $SU(n+1)$ Toda system with single source. Then we discuss applications of this classification result, including construction of the non-topological solutions of A_2, B_2, G_2 Chern–Simons–Higgs system, fully/partially blow-up analysis, and degree-counting formula for mean field Toda system.

On the Correct Solvability of Parabolic Functional Differential Equations with Deviation of Time Argument

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In this paper, we investigate the questions of correct solvability of the initial value problem for a model parabolic differential-difference equation of the form

$$u_t(t,x) = \mathcal{L}u(t,x) + f(t,x),\ t > 0, x \in \mathbb{R}^d, \tag{1}$$

where

$$\mathcal{L}u(t,x) = \Delta u(t,x) + \sum_{k=1}^{N}\{[a_k(u(t+h_k,x))] + i[(\mathbf{b}_k, \nabla u(t+h_k,x))] + [c_k \Delta u(t+h_k,x)]\} - \gamma_0 u(t,x),\ (t,x) \in (0,+\infty)\times\mathbb{R}^d. \tag{2}$$

In equation (2), the Laplace operator Δ acts in $\mathbb{R}^d$ as a linear self-adjoint operator in the space $H = L_2(\mathbb{R}^d)$ with domain $D(\Delta) = W_2^2(\mathbb{R}^d) \subset H$. Coefficients a_k, c_k, h_k, $k = \overline{1,N}$, are real numbers, $h = h_1 < h_2 < ... < h_N$, $h < 0$, and coefficients $\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_N$ are vectors of the Euclidean space $\mathbb{R}^d$.

In this work, we study the problem of finding a function $u :\ (h,+\infty)\times\mathbb{R}^d \to \mathbb{R}$ satisfying equation (1) in $(0,+\infty)\times\mathbb{R}^d$ and the initial condition

$$u\,|_{(h,0]\times\mathbb{R}^d} = \varphi \tag{3}$$

on the set $(h,0]\times\mathbb{R}^d$. Here $\varphi(t,x)$ is a given initial function on the set $(h,0]\times\mathbb{R}^d$.

Let $\mathbf{A}$ be a self-adjoint operator in H bounded from below, $L_{2,\gamma}((a,b),H)$ the Hilbert space of all Bochner-integrable measurable maps of the segment into the space H, endowed with the norm

$$\|f\|_{L_{2,\gamma}} = \left(\int_a^b e^{-2\gamma t}\|f(t)\|_H^2 dt\right)^{1/2},$$

and $W_{2,\gamma}^1((a,b),\mathbf{A})$ the Sobolev space of maps $f \in L_{2,\gamma}((a,b),H)$ whose generalized derivatives $\frac{d}{dt}f$ exist and belong to the space $L_{2,\gamma}((a,b),H)$; the norm in this Sobolev space is given by the equality $\|f\|_{W_{2,\gamma}^1} = [\|f\|_{L_{2,\gamma}}^2 + \|\frac{d}{dt}f\|_{L_{2,\gamma}}^2]^{1/2}$.

Definition. A function $u(t)$ is called a *strong solution of Cauchy problem* (1) with initial condition (3) if it belongs to the space $W^1_{2,\gamma}((h,+\infty),\mathbf{\Delta})$ for some $\gamma\in\mathbb{R}$ and satisfies equation (1) and condition (3).

Theorem 1. *If $h_N \geqslant 0$, then for any sufficiently small coefficients $a_k, \mathbf{b}_k, c_k$, $k\in\overline{1,N}$, there is a finite interval $I=(\alpha,\beta)\subset\mathbb{R}$ such that the following statement holds*:

(S) *If $\gamma\in I$ and $f\in L_{2,\gamma}(\mathbb{R}_+,H)$, then Cauchy problem* (1) *with initial condition* (3) *has a unique solution $u(x,t)$ in the space $W^1_{2,\gamma}(\mathbb{R}_+,\mathbf{\Delta})$, and this solution admits the estimate*

$$\|u\|_{W^1_{2,\gamma}((h,+\infty),\mathbf{\Delta})} \leqslant c[\|f\|_{L_{2,\gamma}((0,+\infty),H} + \|\varphi\|_{W^1_2((h,0),\mathbf{\Delta})}]. \tag{4}$$

If $h_N<0$, then for any coefficients $a_k, \mathbf{b}_k, c_k$, $k\in\overline{1,N}$, there is a infinite interval $I=(\alpha,+\infty)\subset\mathbb{R}$ such that statement (S) *and estimate* (4) *hold.*

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Searching for Mode Propagating in Helical Waveguides

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The method of boundary integral equations is developed for finding the dispersion characteristics of a round waveguide with a deep multistart helical corrugation. This type of waveguides is applied as the space of interaction of radiation with an electron beam in such device of microwave electronics as gyro-TWT.

In case of cylindrical symmetry, electromagnetic field is expressed through electric u and magnetic v Borgnis functions only, which is equivalent to introduction of electric and magnetic Hertz vectors with only nonzero z-components. Thus, the functions u and v satisfy the Helmholtz equation,

$$\Delta u + \varkappa^2 u = 0\,, \quad \Delta v + \varkappa^2 v = 0.$$

Moreover, the functions u, v, and, therefore, all the components of electric and magnetic fields meet the Floquet condition

$$u, v, E_j, H_j(r,\varphi+2\pi l/L, z+l) = \exp(i\pi h_f) u, v, E_j, H_j(r,\varphi,z)\,,\ j=r,\psi,z.$$

The boundary condition on the internal surface of a waveguide is that the tangential component of electric field vanishes on a metal surface.

Introduction of oblique (helical) coordinate system allows one to find the integral representation of Green's function of the helical waveguide [1] and to reduce the problem to the two-dimensional case.

Factorizing with respect to z, we obtain Green's function for the operator obtained as a result of replacement of the "straightened" waveguide.

Their kernels are expressed in terms of Green's function of a general form elliptic operator rather than the Helmholtz equation with the main part being the Laplacian. At the same time, the conditions of the gap for directional derivatives change when crossing the boundary surface.

Vanishing of the tangential component of electric field at the boundary of the waveguide leads to the system of hypersingular boundary integral equations.

We bring the set of equations to a form where the coefficients of the unknown functions and their derivatives are not divided by the function of the boundary, its derivatives, and their combinations. Then these coefficients will contain only sines and cosines of the arguments that are multiples of the number of visits of the screw on the boundary surface.

In this set of equations, both the coefficients and the Fourier components of the unknown functions and their derivatives are numbered by increments equal to the number of visits of the screw on the helical surface. Consequently, the system of linear equations splits into exactly the same number of independent subsystems. Zero initial case corresponds to the subspace in the linear space of exponents of the basis functions for the Fourier transform. Nonzero start determines the corresponding coset in this space.

Finding preconditioning operator relieves the subsystem from hypersingularity and allows one to sufficiently reduce the number of unknown coefficients in a finite Fourier representation.

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Solvability of the Thermo-Viscoelastic Voigt Model[1]

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We consider the following initial-boundary value problem describing the motion of weakly concentrated aqueous solutions of polymers under temperature changes:

$$\frac{\partial v}{\partial t} + \sum_{i=1}^{n} v_i \frac{\partial v}{\partial x_i} - \nu_0 \Delta v - 2\mathrm{Div}\,(\nu_1(\theta)\mathcal{E}(v)) - \varkappa \frac{\partial \Delta v}{\partial t} + \mathrm{grad}\, p(\theta) = f, \quad (1)$$

$$\mathrm{div}\, v = 0, \quad (t,x) \in (0,T) \times \Omega, \quad (2)$$

$$v(0,x) = v_0(x), \quad x \in \Omega, \qquad v|_{[0,T]\times\partial\Omega} = 0, \quad (3)$$

$$\frac{\partial \theta}{\partial t} + \sum_{i=1}^{n} v_i \frac{\partial \theta}{\partial x_i} - \chi \Delta\, \theta = 2(\nu_0 + \nu_1(\theta))\mathcal{E}(v) : \mathcal{E}(v) + 2\varkappa \frac{\partial \mathcal{E}(v)}{\partial t} : \mathcal{E}(v) + g, \quad (4)$$

$$\theta(0,x) = \theta_0(x), \quad x \in \Omega, \qquad \theta|_{[0,T]\times\partial\Omega} = 0. \quad (5)$$

Here v is the vector of velocities, p is the pressure function, θ is the temperature function, $\varkappa > 0$ is the delay time, $\chi > 0$ is the coefficient of thermal conductivity, $\nu_0 > 0$ is the initial coefficient of viscosity, $\nu_1(\theta)$ is a viscosity coefficient, f is the density of external forces, g is an external thermal source, and $\mathcal{E} = (\mathcal{E}_{i,j})$, $\mathcal{E}_{i,j} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$, is the strain rate tensor.

The following result is established.

Theorem. *Let* $\nu_1(\theta) \in C^2(-\infty,+\infty)$, $0 \leqslant \nu_1(\theta) < M$, $M = const$, $f \in L_2(0,T;V^*)$, $v_0 \in V$, $g \in L_1(0,T;H_p^{-2(1-1/p)}(\Omega))$, *and* $\theta_0 \in W_p^{1-2/p}(\Omega)$. *Further, suppose that either* $n = 2$ *and* $1 < p < \frac{4}{3}$ *or* $n = 3$ *and* $1 < p < \frac{5}{4}$. *Then there exists a weak solution to problem* (1)–(5).

Attractors and Pullback-Attractors of Hydrodynamic Equations

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In this report, the theory of attractors is devoted to investigation of the asymptotic behavior as $t \to \infty$ of weak solutions of hydrodynamic equations describing the dynamics of incompressible fluids. Here we have used the concept of a trajectory attractor. It is an effective tool for studying situations where there is no uniqueness of the corresponding boundary value problem solutions. Some examples of a viscoelastic media hydrodynamics for which the existence of attractors is established on the basis of the developed theory are considered. As one of the examples, we present

[1]Joint work with V. P. Orlov (Voronezh State University, Voronezh, Russia).

the existence theorem of attractors for the model of the motion of fluid media with memory.

Let Ω be a bounded domain in the space $\mathbb{R}^n$, $n = 2, 3$. We look at the following initial-boundary value problem:

$$\frac{\partial v}{\partial t} + \sum_{i=1}^{n} v_i \frac{\partial v}{\partial x_i} - \mu_1 \operatorname{Div} \int_0^t e^{-\frac{t-s}{\lambda}} \mathcal{E}(v)(s, Z_\delta(s; t, x))\, ds - \mu_0 \operatorname{Div} \mathcal{E}(v) = -\operatorname{grad} p + f, \quad (1)$$

$$\operatorname{div} v = 0, \quad (t, x) \in (0, +\infty) \times \Omega; \quad v|_{(0,+\infty)\times\partial\Omega} = 0; \quad v(0, x) = v_0(x), \quad x \in \Omega, \quad (2)$$

where $v = (v_1, \dots v_n)$ is the vector-valued velocity function of particles in the fluid.

Theorem. *Suppose the parameters μ_0, μ_1 and λ in* (1)-(2) *are related by $\mu_0 - \mu_1\lambda > 0$ and $f \in V^*$. Then the trajectory space $\mathcal{H}^+$ in problem* (1)-(2) *has a global attractor $\mathcal{A}$. It is bounded in H and compact in V_θ^*; in the topology of V_θ^* it attracts sets of trajectories which are bounded in the norm of $L_\infty(\mathbb{R}_+; H)$.*

In this report, the theory of pullback-attractors of weak solutions for hydrodynamic equations is also considered. This theory is a generalization of the theory of attractors in case of systems with non-autonomous dynamics.

Pullback-attractor is a family of subsets of the phase space of the system: $U = \{\mathcal{U}_\theta\}_{\theta\in\mathbb{R}}$. The set $\mathcal{U}_\theta$ is the pullback-attracting for every moment of an absolute time θ.

This means that if B is a bounded set in the phase space and O is a neighborhood of $\mathcal{U}_\theta$ then values of trajectories at the time θ which start at the points of B belong to the neighborhood O on the assumption of that these trajectories started not earlier than the time $\tau = \tau(B, \theta, O)$.

In the report the existence of minimum pullback-attractors of weak solutions for a model of weakly concentrated aqueous polymer solutions is proved.

Энтропия по Больцману и Пуанкаре

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Уже в 1906 году в работе [1] А. Пуанкаре показал на примерах, что, хотя энтропия сохраняется для уравнения Лиувилля, энтропия среднего по Чезаро больше или равна (как правило, больше) энтропии начального распределения. Эта новая форма H-теоремы была доказана В.В. Козловым и Д. В. Трещевым [2] для уравнения Лиувилля для произвольной гамильтоновой системы. В [3] было показано, что в случае, когда дивергенция скорости динамической системы равна нулю, решение уравнения Лиувилля сходится «туда, куда надо» — средние по Чезаро совпадают с экстремалями по Больцману, т. е. определяются условным принципом максимума энтропии (принципом Больцмана).

Мы доказываем теорему о совпадении среднего по Чезаро с экстремалью по Больцману для уравнения Лиувилля в случае, когда существует положительное стационарное решение уравнения Лиувилля. Мы получили точные формулы, дающие размерность пространства линейных инвариантов для уравнения Лиувилля с дискретным временем для круговой модели Марка Каца, а значит, в силу теоремы о совпадении среднего по Чезаро с экстремалью по Больцману, и размерность пространства стационарных решений.

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О разрешимости задачи радиационно-кондуктивного теплообмена в системе полупрозрачных тел

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Рассматривается начально-краевая задача (здесь $0 < t < T$, T - любое положительное число)

$$c_p \frac{\partial u}{\partial t} - \operatorname{div}(\lambda(u)\nabla u) + 4\pi \int\limits_0^\infty \varkappa_\nu k_\nu^2 h_\nu(u)\, d\nu = \int\limits_0^\infty \varkappa_\nu \int\limits_\Omega I_\nu \, d\omega\, d\nu + f, \quad x \in G, \tag{1}$$

$$\omega \cdot \nabla I + (\varkappa_\nu + s_\nu) I_\nu = s_\nu \mathcal{S}_\nu(I_\nu) + \varkappa_\nu k_\nu^2 h_\nu(u), \quad (\omega, x) \in \Omega \times G, \tag{2}$$

$$\lambda(u)\nabla u \cdot n = 0, \quad x \in \partial G, \tag{3}$$

$$I_\nu|_{\Gamma^-} = \mathfrak{B}_\nu(I_\nu|_{\Gamma^+}), \quad (\omega, x) \in \Gamma^-, \ 0 < \nu < \infty, \tag{4}$$

$$u|_{t=0} = u^0, \quad x \in G, \tag{5}$$

описывающая радиационно-кондуктивный теплообмен в системе $G = \bigcup\limits_{j=1}^m G_j$, состоящей из полупрозрачных тел $G_j \subset \mathbb{R}^3$, разделенных вакуумом. Искомые функции $u(x,t)$, $I_\nu(\omega, x, t)$ имеют физический смысл абсолютной температуры и интенсивности излучения на частоте ν, распространяющегося в направлении $\omega \in \Omega = \{\omega \in \mathbb{R}^3 : \ |\omega| = 1\}$.

Здесь $0 < c_p$, $0 < \lambda(u)$, $0 \leqslant \varkappa_\nu$, $0 \leqslant s_\nu$ и $1 < k_\nu$ — коэффициенты теплоемкости, теплопроводности, поглощения, рассеяния и показатель преломления. Функция $h_\nu(u)$ отвечает спектральному распределению Планка

$$\pi h_\nu(u) = \frac{2\pi\nu^2}{c_0^2} \frac{\hbar\nu}{\exp\left(\hbar\nu/(ku)\right) - 1}$$

при $u > 0$. В уравнении переноса излучения (2) оператор

$$\mathcal{S}_\nu(I_\nu)(\omega, x) = \int\limits_\Omega \theta_\nu(\omega' \cdot \omega) I(\omega', x)\, d\omega'$$

есть оператор рассеяния.

Краевое условие (4) описывает зеркальное отражение и преломление излучения по законам Френеля на границах тел. В нем $\Gamma^- = \{(\omega, x) \in \Omega \times \partial G \mid \omega \cdot n(x) < 0\}$, $\Gamma^+ = \{(\omega, x) \in \Omega \times \partial G \mid \omega \cdot n(x) > 0\}$. Подробное описание условия (4) и доказательство однозначной разрешимости задачи (2), (4) даны в [1].

Работа выполнена при финансовой поддержке РФФИ (грант 13-01-00201) и Совета по грантам при Президенте РФ (проект НШ-2081.2014.1).

В данной работе доказаны существование и единственность обобщенного решения задачи (1)–(5). Установлена теорема сравнения.

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Устойчивость решений начально-краевых задач аэрогидроупругости

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Исследуется устойчивость решений начально-краевых задач для связанных систем интегро-дифференциальных уравнений с частными производными, описывающих динамику деформируемых элементов различных конструкций, находящихся во взаимодействии с газожидкостной средой (обтекаемых потоком жидкости или газа). Принятые в работе определения устойчивости деформируемого (вязкоупругого, упругого) тела соответствуют концепции устойчивости динамических систем по Ляпунову. Изучается устойчивость элементов летательных аппаратов, трубопроводных систем, датчиков измерения параметров газожидкостных сред при различных способах закрепления элементов при дозвуковом или сверхзвуковом режимах обтекания сжимаемой или несжимаемой средой. Воздействие газа или жидкости (в модели идеальной среды) определяется из асимптотических уравнений аэрогидромеханики. Для описания динамики упругих элементов используется как линейная, так и нелинейная теории твердого деформируемого тела.

Для решения связанных задач аэрогидроупругости используется несколько подходов. В частности, один из подходов основан на построении решения аэрогидродинамической части задачи методами теории функций комплексного переменного, при этом аэрогидродинамическая нагрузка (давление жидкости или газа) определяется через функции, описывающие неизвестные прогибы элементов (стержней, пластин, оболочек). Тогда решение исходных задач сводится к исследованию систем связанных интегро-дифференциальных уравнений с частными производными для функций прогибов элементов. Другой подход использует для построения решения аэрогидродинамической части задачи метод Фурье и представление искомых функций (потенциала скорости и прогибов элементов) в виде рядов. При этом аэрогидродинамическая нагрузка также определяется через функции, описывающие неизвестные прогибы элементов, для которых вновь возникает связанная система интегро-дифференциальных уравнений. Исследование устойчивости проводится на основе прямого метода Ляпунова и численных методов. Построены положительно определенные функционалы типа Ляпунова, соответствующие полученным системам интегро-дифференциальных уравнений с частными производными. Проведено также исследование устойчивости на основе численно-аналитических методов (основой которых служит метод Бубнова—Галеркина) с реализацией численного эксперимента. Проведенные численные эксперименты показали удовлетворительное согласование необходимых и достаточных условий устойчивости, полученных численно, с достаточными условиями, полученными аналитически на основе исследования функционалов.

Работа выполнена в рамках государственного задания № 2014/232 Минобрнауки России.

Решение первой начально-краевой задачи для параболических систем в плоских негладких областях

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Рассмотрена первая начально-краевая задача для одномерной (по x) параболической системы в криволинейных областях с негладкими боковыми границами. Методом граничных интегральных уравнений установлена разрешимость (в классическом смысле) этой задачи и получено интегральное представление решения.

Работа второго автора выполнена при частичной финансовой поддержке Совета по грантам при Президенте РФ (проект НШ-2081.2014.1) и гранта РФФИ (проект 13-01-00201-а).

Эффективный метод гармонического отображения сложных областей с приложением к построению расчетных сеток

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Под гармоническим отображением жордановых областей $\mathcal{Z}$ и $\mathcal{W}$ будем понимать гармоническое продолжение $w = \mathcal{F}(z)$ в область $\mathcal{Z}$ заданного гомеоморфизма $\mathcal{B} : \partial\mathcal{Z} \xrightarrow{Hom} \partial\mathcal{W}$ границ этих областей. Согласно теоремам Радо—Кнезера и Шоке [1] достаточным условием осуществления отображением $\mathcal{F} : \mathcal{Z} \xrightarrow{harm} \mathcal{W}$ гомеоморфизма замыканий указанных областей является выпуклость области $\mathcal{W}$. Одним из важных приложений гармонических отображений является построение расчетных сеток, которое осуществляется следующим образом. Выбирая в качестве области $\mathcal{W}$ единичный квадрат $\mathcal{Q} := [0,1]\times[0,1]$ и строя гармоническое отображение $\mathcal{F} : \mathcal{Z} \overset{harm}{\longrightarrow} \mathcal{Q}$, получаем требуемую сетку путем переноса естественной для квадрата (равномерной с шагом h) декартовой сетки в область $\mathcal{Z}$ с помощью этого отображения. При этом граничный гомеоморфизм $\mathcal{B} : \partial\mathcal{Z} \overset{Hom}{\longrightarrow} \partial\mathcal{Q}$ выбирается таким образом, чтобы на прообразах $l_n := \mathcal{B}^{-1}(L_n)$ сторон L_n квадрата $\mathcal{Q}$ «граничная производная» dS/ds отображения $w = \mathcal{B}(z)$ была постоянна, т.е. $dS(w)/ds = |l_n|^{-1}$, $z \in l_n$. Здесь $s(z)$ и $S(w)$ — длины дуг на $\partial\mathcal{Z}$ и на $\partial\mathcal{Q}$. Тогда задача Дирихле для отображения $w = \mathcal{F}(z)$ приобретает вид [2]:

$$\Delta\mathcal{F}(z) = 0, \qquad z \in \mathcal{Z},$$

$$\mathcal{B}(z) = w_n - (i)^{n+1}\,|l_n|^{-1}\big[s(z) - \sigma_n\big], \qquad z \in l_n, \quad n = \overline{1,4};$$

здесь $\sigma_n := \sum_{k=1}^{n-1}|l_k|$, а w_n — вершины квадрата $\mathcal{Q}$. Для решения этой задачи предлагается использовать высокоэффективный аналитико-численный метод мультиполей [3], обладающий экспоненциальной скоростью сходимости. Метод был реализован для построения расчетных сеток в ряде сложных областей и, кроме того, позволил провести теоретическое исследование поведения отображения $\mathcal{F}$ вблизи углов и других геометрических особенностей области $\mathcal{Z}$.

Работа выполнена при финансовой поддержке РФФИ (проект № 13-01-00923), Программы ОМН РАН «Современные проблемы теоретической математики», (проект «Оптимальные алгоритмы решения задач математической физики») и Программы № 3 фундаментальных исследований ОМН РАН.

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О применении принципа усреднения к логистическому уравнению с быстро осциллирующим запаздыванием[1]

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Рассматривается хорошо известное логистическое уравнение с запаздыванием

$$\dot{u} = r(1 - u(t-T))u, \tag{1}$$

[1]Работа выполнена при поддержке проекта № 984 в рамках базовой части государственного задания на НИР ЯрГУ.

где функция $u(t)$ и параметры r и T положительны. Основное предположение состоит в том, что параметр запаздывания T имеет вид

$$T = T_0 + a \operatorname{sign} \sin(\omega t), \qquad T_0 > 0, \qquad a > 0, \qquad \omega \gg 1. \tag{2}$$

Таким образом запаздывание является периодически зависимым и быстро осциллирующим по времени.

Ставится вопрос о локальной — в окрестности состояния равновесия $u_0 \equiv 1$, динамике уравнения (1) при условии (2).

В результате применения метода усреднения удалось получить логистическое уравнение с двумя запаздываниями

$$\dot{v} = r\left[1 - \frac{1}{2}\Big(v(t-h_1) + v(t-h_2)\Big)\right] v.$$

Здесь $h_1 = T_0 - a$, $h_2 = T_0 + a$.

Критерий асимптотической устойчивости состояния равновесия u_0 состоит в выполнении неравенства

$$r < \frac{\pi}{2T_0 \cos \frac{a\pi}{2T_0}} = r_0.$$

При выполнении противоположного строгого неравенства это состояние равновесия неустойчиво.

В критическом случае когда $r = r_0$ показано, что для исходной задачи в окрестности состояния равновесия u_0 имеется двумерное устойчивое локальное инвариантное интегральное многообразие, на котором задача принимает вид обыкновенного дифференциального уравнения для комплексной переменной $\xi = \xi(\tau)$:

$$\frac{d\xi}{d\tau} = \lambda\xi + d|\xi|^2\xi. \tag{3}$$

Здесь $\lambda = \lambda_1(\omega)/\omega + O(\omega^{-2})$, $\lambda_1(\omega)$ — ограниченная функция, а d — некоторая величина, не зависящая от ω, причем $\operatorname{Re} d < 0$.

Решения уравнения (3) и решения из указанного двумерного интегрального многообразия исходного уравнения (1) связаны формулой

$$u(t,\omega) = 1 + \omega^{-1/2}\left(\xi(\omega^{-1}t)\exp\left(i\frac{\pi}{2T_0}t\right) + \bar{\xi}(\omega^{-1}t)\exp\left(-i\frac{\pi}{2T_0}t\right)\right) + O(\omega^{-1}).$$

В силу того, что функция $\operatorname{Re}\lambda_1(\omega)$ является знакопеременной при $\omega \to \infty$, происходит неограниченный процесс «рождения» из состояния равновесия и «гибели» устойчивого цикла.

Нелинейные разностные уравнения, возникающие в задачах о сетях обслуживания

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Работа систем обслуживания и систем связи часто описывается вероятностными моделями.

Для изучения больших систем во многих случаях применяются асимптотические методы и в итоге рассмотрение сводится к изучению детерминированных систем, в частности, систем, заданных дифференциальными уравнениями. При этом представляет интерес поведение решений начально-краевых задач на больших временах, существование и единственность стационарных решений.

Хорошо известный пример нелинейной задачи: на сетке точек $\{i = 0, 1, 2, ...\}$ ищется решение уравнений

$$\frac{du(i,t)}{dt} = \lambda(u^2(i-1,t)-u^2(i,t))-u(i,t)+u(i+1,t),\ u(i,0) = u_i^{(0)}, u(0,t) = 1, \lambda > 0.$$

Мы рассмотрим несколько примеров моделей систем и возникающие для них конечно-разностные уравнения. Специально остановимся на случаях неединственности стационарных решений.

Исследование функций Хи

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Работа посвящена исследованию функций гипергеометрического типа, связанных с решениями модели Калоджеро—Сазерленда. Функция следа этих решений в простом случае является гипергеометрической функцией Гаусса. Обобщение этого результата для функций следа при $n > 2$ привело к новому семейству гипергеометрических функций многих переменных, дифференциальные свойства и интегралы которых описываются интегралами по некоторым траекториям от степенных функций с иррациональными (или комплексными) показателями.

В случае $n = 2$ след может быть представлен с помощью гипергеометрической функции Гаусса

$$E(z_1, z_2) = \frac{z_1^{\lambda_1+1} z_2^{\lambda_2}}{z_1 - z_2} F(\mu + 1, -\mu; \lambda_2 - \lambda_1, \frac{z_2}{z_2 - z_1}),$$

где $\lambda_1, \lambda_2, \mu$ — комплексные константы.

В настоящей работе изучается функция следа при $n = 3$, которая представляется рядом

$$\chi_A(\alpha_1, \alpha_2, \alpha_3; \beta; x, y, z) = \sum_{m,n,k}^{\infty} \frac{(\alpha_1)_{m+n}(\alpha_2 - m)_k(\alpha_3)_{n+k}}{(\beta)_{n+k} m! n! k!} x^m y^n z^k,$$

где $(a)_n = a(a+1)...(a+n-1)$. Далее изучаются аналитические свойства этой функции, которые можно представить следующим образом:

$$\chi_A(\alpha_1, \alpha_2, \alpha_3; \beta; x, y, z) =$$
$$= (1-z)^{-\alpha_2}(1-y)^{-\alpha_1}\chi_A(\alpha_1, \alpha_2, \beta-\alpha_3; \beta; \frac{x(1-z)}{1-y}, \frac{y}{y-1}, \frac{z}{z-1}) =$$
$$= (-x)^{-\alpha_1}(1-z)^{-\alpha_3}\chi_A(\alpha_1, \beta-\alpha_1-\alpha_2, \alpha_3; \beta; \frac{1}{x}, \frac{y}{x(z-1)}, \frac{z}{z-1}) =$$
$$= (1-z)^{\beta-\alpha_2-\alpha_3}(xz-x-y)^{-\alpha_1} \times$$
$$\times \chi_A(\alpha_1, \beta-\alpha_1-\alpha_2, \beta-\alpha_3; \beta; \frac{1}{x+y-xz}, \frac{y}{x+y-xz}, z).$$

Для функции следа при $n = 4$ имеют место аналогичные соотношения.

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О задачах Соболева для особых подмногообразий

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Рассматриваются задачи Соболева для особых подмногообразий. Устанавливается фредгольмовость таких задач и предъявляются соответствующие формулы индекса.

Двойственная формулировка Принципа максимума Понтрягина (ПМП)

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Уже в простейшем конечномерном случае всякое необходимое условие экстремальности, выраженное в инфинитезимальных терминах, в полной мере использует двойственность между дифференциалом исследуемой на экстремум функции и производными по касательным направлениям к многообразию связей при соответствующем значении аргумента. Например, необходимое условие экстремальности значения гладкой функции f в точке $x \in \mathbb{R}^n$ при наличии связей $g_1 = g_2 = 0$ (правило множителей Лагранжа) можно записать в виде соотношения двойственности

$$ker\, df\Big|_x \supset T_x G,$$

где T_xG — касательное пространство в x к $n-2$-мерному многообразию G, заданному уравнениями связей.

В докладе показано на примере типичной задачи оптимального управления — задачи оптимального быстродействия, что и ПМП можно воспринимать как манифестацию двойственности между касательными и соответствующими кокасательными пространствами $T_{x_t}M$ и $T^*_{x_t}M$ к конфигурационному многообразию задачи M вдоль рассматриваемой оптимали $x_t, 0 \leqslant t \leqslant T$. Дадим здесь краткое описание этой связи, начав со следующих обозначений.

Через $\mathcal{L}_X$ обозначим производную Ли над $X \in Vect\, M$ — векторное поле, канонически генерируемое на TM полем X.Через $\mathcal{P}_X$ обозначим двойственное к нему (и однозначно определенное) векторное поле на кокасательном расслоении T^*M, удовлетворяющее соотношению двойственности Э. Картана

$$X < \theta, Y > = < \mathcal{L}_X\theta, Y > + < \theta, \mathcal{P}_X Y > \ \ \forall\theta \in \Lambda^1,\ X, Y \in Vect\, M.$$

Я называю это поле производной Понтрягина. Оно было введено для формулировки принципа максимума Л. С. Понтрягиным как гамильтоново поле на T^*M, индуцированное гамильтонианом H_X на T^*M, канонически определяемым векторным полем X формулой

$$H_X(\psi_u) = < \psi_u, X_u >,\ \ \psi_u \in T^*_u M,\ \ u \in M.$$

Из соотношения двойственности следует, что ограничение поля $\mathcal{P}_X$ с алгебры $C^\infty(T^*M)$ на подмодуль $Vect\, M$ совпадает со стандартным оператором взятия скобок Ли,

$$\mathcal{P}_X H_Y = H_{ad_X Y}\ \forall X, Y \in Vect\, M.$$

Предположим, что оптималь $x_t, 0 \leqslant t \leqslant T$, является траекторией допустимого векторного поля X_t задачи. Используя векторные поля $\mathcal{L}_{X_t}, \mathcal{P}_{X_t}$ и соответствующие им потоки

$$G^{\tau,t}, \Gamma^{\tau,t},\ 0 \leqslant t \leqslant T, \quad G^{\tau,\tau} = id_{TM},\ \Gamma^{\tau,\tau} = id_{T^*M},$$

можно выразить ПМП в форме соотношения двойственной с помощью следующей конструкции.

В каждом касательном пространстве вдоль заданной оптимальной траектории $x_t, 0 \leqslant t \leqslant T$, естественно строится замкнутый выпуклый *конус вариаций* $K_t \subset T_{x_t}M$ с вершиной в начале, который можно переносить с помощью потока $G^{\tau,t}$ в любое другое касательное пространство вдоль x_t, причем

$$G^{\tau,t}K_\tau \subset K_t \text{ при } \tau \leqslant t, \quad Y_t - X_t \in K_t\ \forall t \in [0, T],$$

где Y_t — произвольный допустимый вектор скорости в точке x_t.

Данных определений достаточно, чтобы сформулировать необходимое условие оптимальности траектории $x_t, 0 \leqslant t \leqslant T$, в виде следующего утверждения, основанного на двойственности между касательными пространствами $T_{x_T}M$ и $T^*_{x_T}M$ в конечной точке траектории.

Существует такой ненулевой ковектор $\psi_T \in T^*_{x_T}M$, что отрицательное полупространство его ядра, которое обозначим $ker^{(-)}\psi_T$, содержит конус K_T,

$$ker^{(-)}\psi_T \supset K_T, \qquad (*)$$

т. е. гиперплоскость $\ker\psi_T$ *является опорной к конусу* K_T *и ковектор* ψ_T *направлен в противоположную от* K_T *сторону.*

Наконец, рассмотрим ковекторную функцию

$$\psi_t = (\Gamma^{T,t})^{-1}\psi_T \in T^*_{x_t}M,\ 0 \leqslant t \leqslant T,$$

полученную сносом ковектора ψ_T с помощью потока $(\Gamma^{T,t})^{-1}$ вдоль траектории x_t в обратном направлении. Учитывая двойственность между полями $\mathcal{L}_X$ и $\mathcal{P}_X$, можно из соотношения двойственности (*) легко извлечь условие максимума

$$< \psi_t, X_t >= H_t \geqslant < \psi_t, Y_t > \quad \forall t \in [0,T].$$

Асимптотическое решение задачи о течении жидкости в двумерном канале с малыми периодическими неровностями на стенках

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Мы рассматриваем пример двумерного канала с течением Пуазейля внутри (исключая погранслои), стенки которого заданы уравнениями

$$y = \varepsilon^{4/5}\mu_1\big(x, x/\varepsilon^{2/5}\big) + l,\ y = \varepsilon^{4/5}\mu\big(x, x/\varepsilon^{2/5}\big),$$

где ε — малый параметр, а l — ширина канала. Можно считать, что $\varepsilon = \mathbf{Re}^{-1/2}$, где $\mathbf{Re}$ — число Рейнольдса, или считать этот параметр малой безразмерной вязкостью. Функции $\mu_1(x,\xi)$ и $\mu(x,\xi)$ предполагаются 2π-периодичными по ξ, гладкими и имеющими нулевое среднее.

Как обычно, мы предполагаем, что стационарное течение в канале описывается системой уравнений Навье—Стокса. Граничные условия имеют вид ($\mathbf{U} = (u,v)$ — вектор скорости):

$$\mathbf{U}\big|_{y=\varepsilon^{4/5}\mu_1(x,x/\varepsilon^{2/5})+l} = \mathbf{U}\big|_{y=\varepsilon^{4/5}\mu(x,x/\varepsilon^{2/5})} = \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right).$$

Мы построили асимптотическое решение данной задачи, имеющее двухпалубную структуру (см. [1,2]). При построении осциллирующих решений мы выделяем неосциллирующую (осредненную) часть функций и осцилляции с нулевым средним. В рассматриваемой задаче это разделение является необходимым, так как в одном и том же порядке малости для осциллирующей и осредненной частей скорости краевые условия оказываются различными. Например, в тонком погранслое отсутствуют краевые условия для осциллирующей части, но присутствуют для осредненной. Это, в частности, приводит к некоторым особенностям алгоритма построения решения в пристеночной области.

Также мы провели исследование влияния величины амплитуды неровности и ширины канала на характер течения. В результате мы получили, что при малых амплитудах поток ламинарный, и начиная с некоторого времени — стационарный. При превышении амплитудой некоторого значения A^* в потоке возникают вихри, но исчезают за короткое время, и начиная с некоторого момента времени вихревое течение сменяется ламинарным. При дальнейшим увеличении амплитуды, начиная с некоторого значения A^{**} вначале наблюдается образование вихрей, но в отличии от предыдущих случаев, по истечении некоторого времени течение становится стационарным, но не ламинарным — в «ямке» образуется стационарный вихрь. Аналогично, при заданной амплитуде неровностей, на структуру течения оказывает влияние ширина канала.

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О дестабилизации решений параболических уравнений

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Рассмотрим модельную задачу

$$Lu + q(x)u - u_t = 0, \quad \text{в} \quad \mathbb{R}^N \times (0, \infty), \tag{1}$$

$$u(x, 0) = u_0(x), x \in \mathbb{R}^N, \tag{2}$$

где L — равномерно эллиптический оператор второго порядка с ограниченными коэффициентами (например, $L = \Delta$ — оператор Лапласа), $q(x) \geqslant 0, u_0(x)$ — начальная функция из класса единственности задачи (1), (2). Во многих работах (см., например, [1]) в диссипативном случае $q(x) \leqslant 0$ изучались условия стабилизации к нулю решения задачи (1), (2), т.е. условия существования предела

$$\lim_{t \to \infty} u(x, t) = 0. \tag{3}$$

Будем говорить, что решение задачи (1), (2) дестабилизируется, если существует предел

$$\lim_{t \to \infty} u(x, t) = +\infty \tag{4}$$

равномерно по x на каждом компакте K в $\mathbb{R}^N$.

Дестабилизация, равномерная по x во всем $\mathbb{R}^N$, изучалась в [2].

Пусть

$$q(x) \geqslant \alpha^2 \max(0, \operatorname{sgn}(1-|x|)), \alpha \neq 0. \tag{5}$$

Теорема 1. *Если $N = 1$ или $N = 2$, то для любой непрерывной ограниченной функции $u_0(x) > 0$ решение задачи* (1), (2) *дестабилизируется.*

Теорема 2. *Если $N \geqslant 3$ и постоянная α^2 в* (5) *достаточно большая, то для любой непрерывной положительной функции $u_0(x)$ решение задачи* (1), (2) *дестабилизируется.*

Теорема 3. *Если $N \geqslant 3$ и постоянная α^2 в неравенстве*

$$q(x) \leqslant \alpha^2 \max(0, \operatorname{sgn}(1-|x|)), \alpha \neq 0$$

достаточно мала, то решение задачи (1), (2) *с некоторой непрерывной ограниченной положительной функцией $u_0(x)$ не стабилизируется к нулю и ограничено, т.е. не дестабилизируется.*

В докладе будут указаны и другие ограничения на младший коэффициент $q(x) \geqslant 0$, которые гарантируют дестабилизацию решения задачи (1), (2).

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Принцип максимума в задаче оптимального управления с интегральными уравнениями и фазовыми и смешанными ограничениями

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На фиксированном отрезке времени рассматривается задача с управляемой системой в виде уравнения типа Вольтерры

$$x(t) = x(0) + \int_0^t f(t, s, x(s), u(s))\, ds, \qquad t \in [0, T],$$

фазовыми ограничениями $\Phi_k(t, x) \leqslant 0$, смешанными ограничениями $F_i(t, x, u) \leqslant 0$, $G_j(t, x, u) = 0$, концевыми ограничениями $\eta(x(0), x(T)) = 0$, $\varphi(x(0), x(T)) \leqslant 0$ (также многомерными), и целевым функционалом $J = \varphi_0(x(0), x(T)) \to \min$. Предполагается, что 1) концы исследуемой траектории $x^0(t)$ не лежат на фазовых границах, 2) смешанные ограничения регулярны, т.е. их градиенты по управлению F_{iu}, G_{ju} позитивно-линейно независимы.

Теорема. *Если процесс $(x^0(t), u^0(t))$ доставляет сильный минимум, то найдется нетривиальный набор множителей $a_0 \geqslant 0$, $a \geqslant 0$, b, $\psi(t)$, $\mu_k(t)$, $h_i(t)$, $m_j(t)$, где a_0 — число, a, b — векторы размерностей φ, η, функция $\psi(t)$ размерности n имеет ограниченную вариацию, функции $\mu_k(t)$ не убывают, функции $h_i(t) \geqslant 0$ и $m_j(t)$ измеримы и ограничены, таких, что соответствующая этому набору модифицированная функция Понтрягина*

$$H(s,x,u) = \psi(s) f(s,s,x,u) + \int_s^T \psi(t)\, f_t(t,s,x,u)\, dt,$$

вычисленная вдоль оптимальной траектории $x^0(s)$, удовлетворяет условию максимума:

$$\max_{u \in C(s)} H(s, x^0(s), u) = H(s, x^0(s), u^0(s)) \qquad (\forall s \in [0,T]),$$

где $C(s)$ есть множество всех u, высекаемое смешанными ограничениями $F_i(s, x^0(s), u) \leqslant 0$ и $G_j(s, x^0(s), u) = 0$.

Кроме того, выполнено сопряженное уравнение

$$\dot\psi(s) = -\overline{H}_x(s, x^0(s), u^0(s)),$$

где

$$\overline{H}(s,x,u) = H(s,x,u) - \sum_i h_i(s)\, F_i(s,x,u) - \sum_j m_j(s)\, G_j(s,x,u) - \sum_k \frac{d\mu_k(s)}{dt}\, \Phi(s,x)$$

есть расширенная модифицированная функция Понтрягина, выполнено условие стационарности по управлению $\overline{H}_u(s, x^0(s), u^0(s)) = 0$, а также обычные условия трансверсальности и дополняющей нежесткости.

Доказательство проводится по схеме, изложенной в [1, 2] для задач с ОДУ. Оно использует условия стационарности [3], а также расширение задачи путем перехода к выпуклой правой части управляемой системы с помощью скользящих режимов.

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О некоторых задачах для системы уравнений Пуассона и Стокса

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Рассматриваются краевые задачи для систем уравнений Пуассона и Стокса в областях трехмерного пространства:

$$\begin{aligned} -\Delta u(x) &= h(x), x \in G, & -\Delta u(x) &= h(x), x \in G, \\ (u,n)_\Gamma &= 0, & [u,n]_\Gamma &= 0, \\ \left[\frac{\partial u}{\partial n}, n\right]_\Gamma &= 0 & \left(\frac{\partial u}{\partial n}, n\right)_\Gamma &= 0. \end{aligned}$$

и

$$\begin{aligned} -\Delta u(x) + \nabla p(x) &= h(x), x \in G, & -\Delta u(x) + \nabla p(x) &= h(x), x \in G, \\ (u,n)_\Gamma &= 0, & [u,n]_\Gamma &= 0, \\ \left[\frac{\partial u}{\partial n} - p(x)n, n\right]_\Gamma &= 0 & \left(\frac{\partial u}{\partial n} - p(x)n, n\right)_\Gamma &= 0. \end{aligned}$$

Основной результат — корректность поставленных задач в смысле Адамара—Петровского. Ключевыми моментами доказательства являются аналоги неравенства Фридрихса, адекватные краевым условиям, аналог теоремы Де Рама и разложение пространств Соболева в сумму соленоидальных и потенциальных подпространств. Предполагается обсудить вычислительные аспекты решения указанных задач и физический смысл краевых условий.

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Примеры вычисления мультипликаторов Флоке

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В работе исследуется T-периодическое решение дифференциально-разностного уравнения следующего вида:

$$x'(t) = f(x(t), x(t-1)).$$

Рассматривается проблема построения и использования рациональной аппроксимации для сведения задачи на собственные значения оператора монодромии к краевой задаче для системы обыкновенных дифференциальных уравнений [1]. Таким образом, возникает вопрос о зависимости собственных значений оператора, осуществляющего сдвиг по времени вдоль решений линеаризованного уравнения, от величины этого сдвига [2]. В докладе будут разобраны два примера, показывающие, что указанная зависимость может как являться, так и не являться липшицевой.

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Гамильтонов вид некоторых дифференциальных уравнений в частных производных

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Широко известны гамильтоновы системы обыкновенных дифференциальных уравнений

$$\frac{dx}{dt} = \frac{\partial H(t,x,y)}{\partial y},$$
$$\frac{dy}{dt} = -\frac{\partial H(t,x,y)}{\partial x},$$

где $t \in \mathbb{R}$, $x : \mathbb{R} \to \mathbb{R}^n$, $y : \mathbb{R} \to \mathbb{R}^n$, $H(t,x,y)$ — заданная функция.

Эволюционное дифференциальное уравнение вида

$$\frac{\partial u(t,x)}{\partial t} = J\frac{\delta H(x,u)}{\delta u(x)},$$

где J — гамильтонов оператор [1], а $\frac{\delta H(x,u)}{\delta u(x)}$ — вариационная производная, называется гамильтоновым уравнением. Такие уравнения обладают рядом важных для приложений свойств [1].

Получены проверяемые необходимые и достаточные условия для того, чтобы эволюционное уравнение могло быть записано в гамильтоновом виде.

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Построение систем сравнения в задаче оценивания интегральной воронки динамической системы

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Рассматриваются динамические системы (ДС), описываемые дифференциальными уравнениями в евклидовом пространстве. Под временным сечением интегральной воронки (ИВ) ДС для некоторого начального множества в момент времени t понимается множество точек, в которые можно перейти из начального множества за это время t. Для общих нелинейных систем получение аналитических формул границы временного сечения ИВ является сложной задачей, поэтому естественно рассматривать оценки сверху и снизу этого множества.

Известно, что при получении качественных оценок ИВ решаются (в некотором смысле) разнообразные задачи теории динамических систем, теории устойчивости, теории управления и др.

Используя известные конструкции метода сравнения и теоремы, доказанные автором, конструктивно построены оригинальные системы сравнения (СС) для оценивания ИВ. Для получения аналитических оценок ИВ достаточно проинтегрировать (численно, аналитически) СС только один раз. В докладе эти результаты обобщаются до построения СС, с помощью которых возможно получить оценки ИВ с допустимой (малой) ошибкой. Обсуждаются различные методы получения оценок. Приводятся примеры оценивания ИВ в системах, описывающих движение ЛА.

Сингулярные функционально-дифференциальные уравнения: разрешимость, число решений, асимптотики решений

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В докладе рассматриваются задачи Коши вида

$$\alpha(t)\,x'(t) = f\left(t, x(t), x(g(t)), x'(t), x'(h(t))\right), \tag{1}$$

$$x(0) = 0, \tag{2}$$

где $x : (0,\tau) \to \mathbb{R}$ — неизвестная функция, $f : D \to \mathbb{R}$ — непрерывная функция, $D \subset (0,\tau) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, $\alpha : (0,\tau) \to (0,+\infty)$, $g : (0,\tau) \to (0,+\infty)$, $h : (0,\tau) \to (0,+\infty)$ — непрерывные функции, $g(t) \leqslant t$ и $h(t) \leqslant t$ при $t \in (0,\tau)$, $\alpha(t) \to 0$ при $t \to +0$, причем либо

$$\frac{\alpha(t)}{t} \to \sigma,\ t \to +0,$$

где $0 \leqslant \sigma < +\infty$, либо

$$\frac{\alpha(t)}{t} \to +\infty,\ t \to +0.$$

Решением задачи (1), (2) называется непрерывно дифференцируемая функция $x : (0, \rho) \to \mathbb{R}, 0 < \rho < \tau$, которая тождественно удовлетворяет уравнению (1) при всех $t \in (0, \rho)$ и при этом $x(t) \to 0, t \to +0$.

Отдельно рассматриваются линейная и возмущенная линейная задачи (1), (2).

Формулируются достаточные условия, при выполнении которых у задачи (1), (2) существует непустое множество решений $x : (0, \rho) \to \mathbb{R}$ (ρ достаточно мало) с определенными свойствами. Исследуется вопрос о числе таких решений.

О решениях гибридных систем функционально-дифференциальных уравнений

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В докладе рассматриваются задачи Коши

$$\alpha(t) x'(t) = f(t, x(g(t)), x'(h(t))), \tag{1}$$

$$x(0) = col(0, 0), \tag{2}$$

где $x : (0, \tau) \to \mathbb{R}^2$ — неизвестная функция, $f : D \to \mathbb{R}^2$ — непрерывная функция, $D \subset (0, \tau) \times \mathbb{R}^2 \times \mathbb{R}^2$, $g : (0, \tau) \to (0, +\infty), h : (0, \tau) \to (0, +\infty)$ — непрерывные функции, $g(t) \leqslant t, h(t) \leqslant t$, $t \in (0, \tau)$, 2×2-матрица $\alpha(t)$ определена следующим образом: либо

$$\alpha(t) = \begin{pmatrix} \alpha_1(t) & 0 \\ 0 & \alpha_2(t) \end{pmatrix},$$

либо

$$\alpha(t) = \begin{pmatrix} \alpha_1(t) & 0 \\ 0 & 1 \end{pmatrix},$$

где $\alpha_i : (0, \tau) \to (0, +\infty)$ — непрерывные функции, $\lim\limits_{t \to +0} \alpha_i(t) = 0, i \in \{1, 2\}$.

Решением задачи (1), (2) называется непрерывно дифференцируемая функция $x : (0, \sigma) \to \mathbb{R}^2, 0 < \sigma < \tau$, которая тождественно удовлетворяет уравнению (1) при всех $t \in (0, \sigma)$ и при этом

$$\lim_{t \to +0} x(t) = col(0, 0).$$

Указываются достаточные условия, при которых задача (1), (2) имеет непустое множество решений $x : (0, \sigma) \to \mathbb{R}^2$ (σ достаточно мало), определены свойства каждого из этих решений и их первых производных при $t \in (0, \sigma)$. Рассмотрен вопрос о количестве указанных решений.

Сходимость равномерного аттрактора явной спектрально-разностной схемы для двухслойной квазигеострофической модели общей циркуляции атмосферы

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Рассматривается двухслойная квазигеострофическая модель общей циркуляции атмосферы, основными переменными которой являются баротропная и бароклинная составляющие функции тока [1]. Предполагается, что правые части уравнений модели зависят от времени и существенно ограничены по норме квадратично интегрируемыми на сфере функциями при $t \in [0, +\infty)$. Множество всех возможных правых частей строится так, чтобы операторы положительных сдвигов по времени переводили его в себя. Рассматриваемая система имеет равномерный (относительно правых частей) компактный аттрактор [2]. Дискретизация модели по пространственным переменным проводится по методу Галеркина с проектированием уравнений на объединение собственных подпространств оператора Лапласа—Бельтрами, отвечающих собственным значениям с номерами от 1 до N. Для аппроксимации по времени используется явная конечно-разностная схема с постоянным шагом сетки τ. Предполагается, что при изменении шага по времени и размерности инвариантного подпространства выполняется неравенство $\tau^{2/3}N(N+1) \leqslant C$ с некоторой константой C. Для дискретной задачи вводятся замкнутые шары такие, что: 1) при достаточно малых τ решение, начатое внутри шара, всегда остается в нем; 2) радиус шара стремится к бесконечности при $\tau \to 0$. Априорные оценки решения показывают, что сужение динамических операторов схемы на упомянутые шары обладает компактным равномерно притягивающим множеством. На основании теории семейств полупроцессов с дискретным временем [3] устанавливается, что при достаточно малых τ явная схема имеет локальный равномерный аттрактор, область притяжения которого бесконечно расширяется при $\tau \to 0$. Исследование сходимости аттрактора схемы сводится к проверке выполнения условий теоремы о полунепрерывной сверху зависимости от параметра равномерных аттракторов семейств полупроцессов [4]. Доказано, что при $\tau \to 0$ и $N \to \infty$ равномерные аттракторы схемы лежат в сколь угодно малой окрестности истинного аттрактора модели.

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Корпоративная динамика сильно связанных распределенных систем

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Рассматривается вопрос о динамике двух нелинейных систем с распределенными параметрами

$$\begin{aligned} \dot{u}_1 &= F_1(u_1) + K\left[\int\limits_{-\infty}^{\infty} \Phi(s) u_2(t, x+s)\, ds - u_1\right], \\ \dot{u}_2 &= F_2(u_2) + K\left[\int\limits_{-\infty}^{\infty} \Phi(s) u_1(t, x+s)\, ds - u_2\right] \end{aligned} \tag{1}$$

с периодическими краевыми условиями

$$u_j(t, x+2\pi) \equiv u_j(t, x) \quad (j = 1, 2). \tag{2}$$

Здесь $u_{1,2} \in \mathbb{R}^n$, $F_{1,2}(u)$ — достаточно гладкие нелинейные вектор-функции, K — положительный параметр, характеризующий степень связи между двумя системами, а функция $\Phi(s)$ имеет вид

$$\Phi(s) = \frac{1}{d\sqrt{\pi}} \exp\left(-\frac{(s+h)^2}{d^2}\right) \quad (d > 0). \tag{3}$$

Основное предположение настоящей работы, открывающее путь к применению асимптотических методов, состоит в том, что параметр K предполагается достаточно большим: $K \gg 1$, т. е. связь является «сильной», а коэффициент d является достаточно малым:

$$0 < d \ll 1,$$

т. е. носитель функции Φ сосредоточен, в основном, в некоторой достаточно малой окрестности точки числовой оси.

При этих условиях рассмотрим вопрос о поведении всех решений краевой задачи (1), (2) с начальными условиями из произвольной (ограниченной при $K \to \infty$, $d \to 0$) области фазового пространства $C_{[0,2\pi]}(\mathbb{R}^n) \times C_{[0,2\pi]}(\mathbb{R}^n)$.

В работе показано, что динамика системы (1), (2) при условии $K \gg 1$, $d \ll 1$ описывается нелокальной динамикой семейства нелинейных краевых задач параболического типа. Отсюда следует, что в ней могут наблюдаться сложные по структуре колебания, и оказывается характерным явление мультистабильности.

К принципу максимума Понтрягина для систем с последействием

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Обсуждается подход к разработке теории принципа максимума Понтрягина для систем с последействием, основанный на методологии и конструкциях i-гладкого анализа [4–6]. Подход позволяет в частности установить, аналогично конечномерному случаю, связь принципа максимума Понтрягина с методом динамического программирования в предположении инвариантной дифференцируемости соответствующих функционалов.

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Метод направляющих функций в исследовании асимптотического поведения решений дифференциальных включений[1]

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В настоящем докладе предлагается использовать негладкую направляющую функцию для исследования асимптотического поведения решений дифференциальных включений следующего вида:

$$x'(t) \in F(t, x(t)), \quad t \in \mathbb{R}, \tag{1}$$

в предположении, что мультиотображение $F : \mathbb{R} \times \mathbb{R}^n \multimap \mathbb{R}^n$ имеет выпуклые компактные значения и удовлетворяет верхним условиям Каратеодори (по поводу терминологии см., например, [1]).

[1]Работа выполнена при финансовой поддержке РФФИ (проект №№ 12-01-00392, 14-01-00468).

Определение 1. Локально липшицева функция $V : \mathbb{R}^n \to \mathbb{R}$ называется прямым потенциалом, если существует такое число $r_\upsilon > 0$, что соотношение $\langle \upsilon, \tilde{\upsilon} \rangle > 0$ выполнено для всех $\upsilon, \widetilde{\upsilon} \in \partial V(x)$, $\|x\| \geqslant r_\upsilon$, где $\partial V : \mathbb{R}^n \multimap \mathbb{R}^n$ — субдифференциал функции V (см., например, [2]).

Пусть $g : \mathbb{R} \to \mathbb{R}$ — четная непрерывно дифференцируемая функция, для которой $\inf\{g(t),\ t \in R\} \geqslant 1$.

Определение 2. Прямой потенциал $V : \mathbb{R}^n \to \mathbb{R}$ называется направляющим потенциалом для включения (1) вдоль функции $g(\cdot)$, если найдется $r_0 \geqslant r_\upsilon$ такое, что

$$\langle \upsilon, g'(t)x + g(t)y \rangle \geqslant 0, \quad \text{почти для всех } t > 0,$$

$$\langle \upsilon, g'(t)x + g(t)y \rangle \leqslant 0, \quad \text{почти для всех } \ t < 0,$$

для всех $y \in F(t,x),\ \upsilon \in \partial V(g(t)x),\ \|x\| \geqslant r_0$.

Пусть регулярная функция $V : \mathbb{R}^n \to \mathbb{R}$ является направляющим потенциалом для включения (1) вдоль функции $g(\cdot)$.

Теорема. *Если выполнено условие* $\lim_{\|x\| \to +\infty} V(x) = -\infty$, *то каждое решение включения* (1) *с начальным условием* $x(0) = x_0$ *удовлетворяет оценке*

$$\|x(t)\| \leqslant k \cdot \frac{1}{g(t)}$$

для некоторого $k > 0, t \in \mathbb{R}$.

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Классические решения методом характеристик граничных задач для гиперболических уравнений

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Авторами получены классические решения для многих граничных задач и задач с интегральными условиями для дифференциальных уравнений с частными производными. В аналитическом виде выписаны решения в случае двух независимых переменных для:

- задачи Коши для уравнения гиперболического типа с постоянными коэффициентами, оператор которого представляет композицию операторов первого порядка;
- смешанных задач для гиперболических уравнений второго порядка;

- смешанных задач для биволнового уравнения;
- смешанных задач для волнового уравнения с интегральными граничными условиями;
- задачи для волнового уравнения с интегральными условиями в области искомого решения;
- задачи управления условиями Коши или граничными условиями, если значения искомой функции или ее производных заданы внутри области искомого решения.

Показано, что классические решения во всей области задания уравнения зависят от однородных условий согласования на заданные функции задачи в общих угловых точках соприкосновения, являющихся необходимыми и достаточными. Если эти условия неоднородны, то необходимо корректно ставить задачи по другому, например, рассматривать задачи сопряжения с условиями сопряжения на соответствующих характеристиках, расположенных внутри области задания искомой функции.

С другой стороны, как известно, на основе классических решений рассматриваемых граничных и других задач строятся численные методы решения этих задач. В литературе по численным методам, как правило, не делается акцент на важность учета условий согласования на заданные функции для полной и правильной постановки корректной рассматриваемой задачи и использования соответствующих численных методов.

Для примера рассмотрим постановку первой смешанной задачи для уравнения Клейна—Гордона—Фока в криволинейной полуполосе, для которой построено классическое решение. Задача рассматривается на плоскости $\mathbb{R}^2$ двух независимых переменных t и x относительно выбранной декартовой системы координат. В криволинейной полуполосе $Q \subset \mathbb{R}^2$ рассмотрим уравнение Клейна—Гордона—Фока в одномерном случае вида

$$\partial_{tt}u - a^2\partial_{xx}u - \lambda(t,x)u = f(t,x), \quad (t,x) \in Q,$$

где λ, f — заданные функции $\lambda, f : (t,x) \in \bar{Q} \to \lambda(t,x),\ f(t,x) \in \mathbb{R}$, $\bar{Q}$ — замыкание области Q, a^2 — положительное действительное число, ∂_{tt}, ∂_{xx} — частные производные и $\partial_{tt} = \frac{\partial^2}{\partial t^2}$, $\partial_{xx} = \frac{\partial^2}{\partial x^2}$. Здесь область Q ограничена нижним основанием — линией AB с уравнением $t = \nu(x)$, боковыми линиями $x = \gamma^{(j)}(t)$, $j = 1,2$, $\gamma^{(1)}(t) < \gamma^{(2)}(t)$ для всех $t \in [t_{sj}, \infty)$. Здесь $t_{s1} = \nu(x_0)$, $t_{s2} = \nu(x_{-1})$. В дальнейших рассуждениях будет удобно ввести некоторую переиндексацию, а именно: $\gamma^{(0)} = \gamma^{(1)}$, $\gamma^{(-1)} = \gamma^{(2)}$.

Условие. *Линия $t = \nu(x)$ определена для всех $x \in [x_0, x_{-1}]$, $x_0 < x_{-1}$, функция $\nu : x \in [x_0, x_{-1}] \to \nu(x) \in \mathbb{R}$ непрерывно дифференцируема, т.е. $\nu \in C^1[x_0, x_{-1}]$, производная $\nu'(x)$ удовлетворяет условию $|\nu'(x)| < \frac{1}{a}$ для всех $x \in [x_0, x_{-1}]$. Аналогично, $\gamma^{(j)} \in C^1[t_{sj}, \infty)$, $j = 1,2$, и выполены неравенства $\left|{\gamma^{(j)}}'(t)\right| < a$ для всех $t \in [t^{(j)}, \infty)$.*

Обозначим через $\tilde{\gamma}^{(j)}$ продолжения функций $\gamma^{(j)}$ на все множество $\mathbb{R}$ таким образом, что $\tilde{\gamma}^{(1)} \in C^1(\mathbb{R})$ и $\tilde{\gamma}^{(1)}(t) < \tilde{\gamma}^{(2)}(t)$ для всех $t \in \mathbb{R}$. Аналогично, $\tilde{\nu}$ — продолжение на $\mathbb{R}$ функции ν, где $\tilde{\nu} \in C^1(\mathbb{R})$.

Таким образом,

$$Q = \Big\{ (t,x) \in \mathbb{R}^2 \big| \, t \in \mathbb{R}, x \in \Big(\tilde{\gamma}^{(1)}(t), \tilde{\gamma}^{(2)}(t) \Big) \Big\} \cap \big\{ (t,x) \in \mathbb{R}^2 \big| \, x \in \mathbb{R}, t > \tilde{\nu}(x) \big\},$$
$$\bar{Q} = \Big\{ (t,x) \in \mathbb{R}^2 \big| \, t \in \mathbb{R}, x \in \Big[\tilde{\gamma}^{(1)}(t), \tilde{\gamma}^{(2)}(t) \Big] \Big\} \cap \big\{ t,x) \in \mathbb{R}^2 \big| \, x \in \mathbb{R}, t \geqslant \tilde{\nu}(x) \big\}.$$

К уравнению присоединяются начальные условия

$$u(\nu(x),x)\varphi(x), \quad \partial_t u(\nu(x),x) = \psi(x), \quad x \in [x_0, x_{-1}]$$

и граничные условия

$$u(t,\gamma^{(1)}(t)) = \mu^{(1)}(t), t \in [\nu(x_0), \infty),$$
$$u(t,\gamma^{(2)}(t)) = \mu^{(2)}(t), t \in [\nu(x_{-1}), \infty).$$

Бифуркационные задачи для одного из модельных уравнений нанотехнологий

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Одной из математических моделей процесса формирования рельефа на поверхности плоских подложек служит следующее дифференциальное уравнение с частными производными с отклоняющимся аргументом [1]. В перенормированном виде это уравнение в одном из частных случаев можно записать как

$$u_t = au_{xx} + hw_x + b_1hw_x^2 + b_2hw_x^3, \tag{1}$$

где $a > 0, b_1, b_2 \in R, |h| < 1, w = u(t, x+h), u = u(t,x)$. Уравнение (1) рассмотрено вместе с периодическими краевыми условиями

$$u(t, x+2\pi) = u(t,x), u_x(t,0) = u_x(t,2\pi), \tag{2}$$

которые достаточно традиционны при рассмотрении данных задач. Уравнение (1) можно также изучить вместе с краевыми условиями

$$u_x(t,0) = u_x(t,2\pi), u_{xx}(t,0) = u_{xx}(t,2\pi). \tag{3}$$

Последний вариант даже более осмыслен с физической точки зрения.

Обратимся сначала к более классическому варианту краевой задачи, т. е. краевой задаче (1), (2). Здесь можно показать, что при $a > a_{\text{кр}}$ однородные состояния равновесия краевой задачи (1), (2) устойчивы в смысле определения Ляпунова, и неустойчивы, если $a \in (0, a_{\text{кр}})$. Если положить $a = a_{\text{кр}}(1+\gamma\varepsilon)$, $\gamma = \pm 1$, $0 < \varepsilon << 1$, то для краевой задачи (1), (2) можно найти пространственно неоднородные решения, в том числе и устойчивые по Ляпунову. Найденные решения описывают волновой нанорельеф на поверхности плоской мишени под воздействием потока ионов. Для соответствующих решений выписаны асимптотические формулы. Обоснование результатов основано на применении аппарата теории нормальных форм, метода интегральных многообразий в сочетании с асимптотическими методами. Часть результатов опубликована в работах [2–5].

Рассмотрена также и бифуркационная задача при втором варианте выбора краевых условий, т. е. краевых условий (3). Для краевой задачи (1), (3) удается показать существование локального аттрактора, все решения на котором неустойчивы.

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Спектральный критерий устойчивости и задача Коши для уравнения Хилла с затуханием при параметрическом резонансе

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Во втором приближении асимптотического метода усреднения решается уравнение Хилла

$$\ddot{z} + \alpha\dot{z} + \{p_0 + \sum_{n=1}^{\infty}[p_n\cos(nt) + q_n\sin(nt)]\}z = 0$$

с начальными условиями $z(0) = z_0$, $\dot{z}(0) = \dot{z}_0$ и $|p_n|$, $|q_n|, \alpha < 1$.

Если $\lambda < 0$, где λ выражается через коэффициенты ряда Фурье p_0, p_n, q_n, то имеем неустойчивость тривиального решения уравнения (параметрический резонанс). При $p_0 \approx 0$ (0-я область резонанса), $\lambda = 4p_0 + \sum_{j=0}^{3} Q_j$, где

$$Q_0 = 2\sum_n \frac{p_n^2 + q_n^2}{n^2}, \quad Q_1 = \sum_{n,n_1;n>n_1} \frac{(p_n p_{n_1} + q_n q_{n_1})p_{n-n_1} + (q_n p_{n_1} - p_n q_{n_1})q_{n-n_1}}{n^2(n-n_1)^2},$$

$$Q_2 = \sum_{n,n_1;n_1>n} \frac{(p_n p_{n_1} + q_n q_{n_1})p_{n_1-n} + (p_n q_{n_1} - q_n p_{n_1})q_{n_1-n}}{n^2(n_1-n)^2},$$

$$Q_3 = \sum_{n,n_1} \frac{(p_n p_{n_1} - q_n q_{n_1})p_{n+n_1} + (q_n p_{n_1} + p_n q_{n_1})q_{n+n_1}}{n^2(n+n_1)^2}.$$

При $p_0 \approx 1/4$ (1-я область резонанса), $p_0 \approx 1$ (2-я область) и $p_0 \approx 9/4$ (3-я область), имеем $\lambda = s^2 - r^2 - \beta^2 + \alpha^2$, где для 1-й области

$$s = 2\sqrt{p_0} - 1 - \frac{1}{8p_0}\sum_{n,n\neq 1}\frac{p_n^2+q_n^2}{2\sqrt{p_0}-n} - \frac{1}{8p_0}\sum_n \frac{p_n^2+q_n^2}{2\sqrt{p_0}+n} - \frac{\alpha^2}{4\sqrt{p_0}},$$
$$r = \frac{p_1}{2\sqrt{p_0}} + \frac{1}{4p_0}\sum_n \frac{p_n p_{n+1} + q_n q_{n+1}}{n(n+1)} + \frac{\alpha q_1}{2\sqrt{p_0}},$$
$$\beta = \tfrac{q_1}{2\sqrt{p_0}} + \tfrac{1}{4p_0}\textstyle\sum_n \tfrac{p_n q_{n+1} - q_n p_{n+1}}{n(n+1)} - \tfrac{\alpha p_1}{2\sqrt{p_0}},$$

а для 2-й области

$$s = 2\sqrt{p_0} - 2 - \frac{1}{8p_0}\sum_{n,n\neq 2}\frac{p_n^2+q_n^2}{2\sqrt{p_0}-n} - \frac{1}{8p_0}\sum_n \frac{p_n^2+q_n^2}{2\sqrt{p_0}+n} - \frac{\alpha^2}{4\sqrt{p_0}},$$
$$r = \frac{p_2}{2\sqrt{p_0}} + \frac{q_1^2 - p_1^2}{4p_0} + \frac{1}{2p_0}\sum_n \frac{p_n p_{n+2} + q_n q_{n+2}}{n(n+2)} + \frac{\alpha q_2}{4\sqrt{p_0}},$$
$$\beta = \frac{q_2}{2\sqrt{p_0}} - \frac{p_1 q_1}{2p_0} + \frac{1}{2p_0}\sum_n \frac{p_n q_{n+2} - q_n p_{n+2}}{n(n+2)} - \frac{\alpha p_2}{4\sqrt{p_0}}.$$

Аналогичные формулы получаются для 3-й области. При $\lambda > \alpha^2$ имеем быстрые колебания составляющих $z(t)$ с затухающими медленно колеблющимися амплитудами. Если $0 < \lambda < \alpha^2$, то амплитуды быстрых колебаний экспоненциально затухают без медленных осцилляций.

Во всех случаях получены решения задачи Коши. Результаты теории проверялись на многочисленных примерах. В отсутствие затухания ($\alpha = 0$) задача Коши решена в работе [1].

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Сравнение двух методов решения обратных задач вариационного исчисления[1]

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При решении различных классов обратных задач вариационного исчисления, заключающихся в отыскании функционала, для которого данное уравнение является уравнением Эйлера, в основном используется интегрирование по вспомогательному параметру соответствующей уравнению билинейной формы (см., например, [1, 2]). Другой подход к решению обратных задач вариационного исчисления для некоторых типов задач предложен в [3]. Он основан на интегрировании по части переменных функций, стоящих перед производными второго порядка в данном уравнении. Для обыкновенных дифференциальных уравнений второго порядка с

[1]Совместная работа с В. Г. Задорожним.

отклоняющимися аргументами этот подход применялся в [4]. Используя второй подход, при некоторых условиях получено решение обратной задачи вариационного исчисления для дифференциального уравнения второго порядка с частными производными и отклоняющимися аргументами.

Производится сравнение двух методов решения обратных задач вариационного исчисления для обыкновенных дифференциальных уравнений второго порядка, обыкновенных дифференциальных уравнений второго порядка с отклоняющимися аргументами, дифференциальных уравнений второго порядка с частными производными и отклоняющимися аргументами. Приводятся примеры задач, где оба метода дают одинаковый результат. Также рассматриваются задачи, решение которых легко может быть найдено с использованием второго метода, а при помощи первого метода решение в явном виде найти невозможно. Например, для уравнения

$$z_{xx}(x,y)+z_{xy}(x,y)+z_{yy}(x,y)+\exp\left(z(x,y)z(\omega(x,y))\right)+\\+\exp\left(z(x,y)z(\gamma(x,y))\right)(z(x,y)z(\gamma(x,y))-1)/z^2(x,y)=0,$$

где дважды непрерывно дифференцируемые функции z определены в открытом круге Q с центром в начале координат, значения функций z на границе Q заданы, ω — поворот на некоторый угол, γ — обратное к ω отображение, используя второй метод, получаем решение обратной задачи вариационного исчисления в виде

$$\int\limits_Q(-(z_x^2(x,y)+z_x(x,y)z_y(x,y)+z_y^2(x,y))/2+\exp\left(z(x,y)z(\omega(x,y))\right)/z(\omega(x,y)))dxdy.$$

Применяя же первый традиционный подход, приходим к необходимости вычисления неберущегося интеграла

$$\int\limits_Q\int\limits_0^1(s(z_{xx}(x,y)+z_{xy}(x,y)+z_{yy}(x,y))+\exp\left(s^2z(x,y)z(\omega(x,y))\right)+\\+\exp\left(s^2z(x,y)z(\gamma(x,y))\right)(s^2z(x,y)z(\gamma(x,y))-1)/(s^2z^2(x,y)))z(x,y)dsdxdy.$$

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О некоторых задачах динамической реконструкции и устойчивого управления системами с запаздыванием

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В докладе обсуждаются два типа задач — задачи динамического восстановления неизвестных входных воздействий (задачи динамической реконструкции), а также задачи управления при наличии неконтролируемых возмущений. В качестве объекта исследований взяты системы дифференциальных уравнений с последействием. Приводятся алгоритмы решения указанных задач, которые являются устойчивыми к информационным помехам и погрешностям вычислений. Алгоритмы, ориентированные на компьютерную реализацию, позволяют осуществлять процесс решения в темпе «реального» времени. При этом они учитывают неточные измерения фазовых траекторий и являются регулирующими в том смысле, что конечный результат тем лучше, чем точнее поступающая информация. В основе предлагаемых алгоритмов лежит известный в теории гарантированного управления метод экстремального сдвига.

Содержание первой из рассматриваемых задач следующее. Задан объект, описываемый системой дифференциальных уравнений с запаздыванием. Его решение зависит от изменяющегося во времени входного воздействия, которое можно трактовать как управление. Как это управление, так и решение системы заранее не заданы, но, возможно, известно множество, ограничивающее допустимую реализацию входного воздействия. В процессе функционирования системы измеряются (с ошибкой) ее фазовые состояния. Необходимо сконструировать алгоритм приближенного восстановления ненаблюдаемой «части» координат, а также входного воздействия, обладающий свойствами динамичности и устойчивости.

Суть второй задачи такова. Имеется две системы, функционирующие на заданном конечном промежутке времени. Предполагается, что на одну систему (назовем ее эталонной) действует неконтролируемое возмущение. Априорная информация о каких либо ограничениях (мгновенных или интегральных) на это возмущение может отсутствовать. В частности, может быть известно лишь, что возмущение является измеримой (по Лебегу) функцией, принадлежащей пространству функций, интегрируемых с квадратом евклидовой нормы. Кроме того, имеется еще одна управляемая система (назовем ее реальной). Требуется построить закон управления по принципу обратной связи реальной системой, обеспечивающий равномерную «близость» решения этой системы к решению эталонной системы.

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Постановка задачи выбора схемно-технических решений десантного аппарата для доставки автоматического планетохода

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Представляется подход к решению задачи выбора схемно-технических решений перспективного десантного аппарата (ДА), состоящего из посадочного аппарата (ПА), автоматического планетохода (ПХ) и системы ввода его в действие (СВД), для обеспечения максимума массы полезной нагрузки ПХ. Проведен анализ факторов, влияющих на принятие решений, исследованы закономерности комплексного функционирования систем на различных этапах функционирования ДА.

Задача имеет ряд особенностей:

1. Каждая составная часть ДА имеет собственные служебные и специальные системы, которые на разных этапах функционирования выполняют аналогичные функции. Например, служебные системы обеспечения электроснабжения, управления, связи, теплового режима составных частей ДА могут быть едиными или частично взаимозаменяемыми на отдельных этапах его функционирования. Конструкция, комплекс управления и ходовая часть ПХ могут совмещать функции обеспечения ввода в действие ПХ, то есть участвовать в посадке, отделении, развертывании и вводе на поверхность. При принятии решений по совмещению функций систем составных частей ДА необходимо учитывать изменения надежности и риска при выполнении задачи.

2. При выборе параметров и схемно-технических решений одной системы составной части ДА накладываются ограничения на выбор параметров и схемно-технических решений другой системы. Например, выбор схемы ввода в действие планетохода накладывает ограничения на допустимые диапазоны массово-габаритных параметров планетохода. Также имеет место функциональная зависимость показателей качества от схемно-технических решений на основе статистических и эвристических оценок. Исходя из анализа факторов, влияющих на принятие решений, в общем виде сформулирована многокритериальная задача выбора схемно-технических решений ДА с ПХ.

О достаточных условиях устойчивости периодических решений нелинейных систем с двумя малыми параметрами

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Рассматривается система дифференциальных уравнений

$$\begin{cases} \dot{x_1} = f_1(x_1) + \mu_1\gamma_1(t, x_2), \\ \dot{x_2} = f_2(x_2) + \mu_2\gamma_2(t, x_1). \end{cases} \tag{1}$$

Предполагается, что функции $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}^n, \gamma_1, \gamma_2 : \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^n$ имеют непрерывные производные по соответствующим пространственным переменным x_1, x_2. Функции γ_1, γ_2 являются T-периодическими функциями по первой переменной.

При нулевых значениях параметров μ_1 и μ_2 система (1) распадается на два автономных уравнения

$$\dot{x_i} = f_i(x_i), i = 1, 2, \tag{2}$$

каждое из которых имеет T-периодические решение $\varphi_i(t)$. Также предполагается, что число 1 является простым собственным значением у операторов сдвига по траектории линеаризованных на φ_i уравнений (2).

Найдены условия устойчивости периодических решений системы (1).

Теоремы о дифференциальных неравенствах для сингулярных функционально-дифференциальных уравнений

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Пусть $1 < p < \infty$; $p' = \dfrac{p}{p-1}$. Определим пространство D^p абсолютно непрерывных функций $x\colon [0,b] \to \mathbb{R}$ таких, что $\dot{x} \in L^p$. Также определим пространство $D^pS(\tau)$ [1, С. 125] абсолютно непрерывных на $[0,\tau)$ и $[\tau, b]$ функций $y\colon [0,b] \to \mathbb{R}$ таких, что $\dot{y} \in L^p$ и $\bigl|y(\tau) - y(\tau - 0)\bigr| < \infty$. Положим $u^s(t) = u(t)\theta(t-s)$, где $\theta(\tau)$ — функция Хевисайда.

Рассмотрим полуоднородную задачу Коши

$$\begin{cases} (\mathcal{L}x)(t) \equiv \dot{x}(t) + \dfrac{k}{t} x\left(\dfrac{t}{\rho}\right) + (Tx)(t) = f(t), \qquad t \in [0,b], \\ x(0) = 0. \end{cases} \tag{1}$$

Здесь $\rho \geqslant 1$, $k \in \mathbb{R}$; оператор $T\colon D^p \to L^p$ вполне непрерывен и вольтерров.

Для задачи (1) доказаны теоремы о дифференциальных неравенствах, являющиеся обобщением результатов работы [2]. С помощью этих теорем получены двусторонние оценки функции Коши $C(t,s)$ задачи (1) и эффективные признаки положительности оператора Коши. Приведем примеры таких теорем и оценок, получаемых с их помощью.

Теорема 1. *Пусть оператор T изотонен и существует функция $u > 0$ такая, что $u^s \in D^pS(s)$ для любого $s \in (0,b)$ и $\mathcal{L}u \leqslant 0$.*

Тогда при $t \geqslant s > 0$ справедлива оценка $C(t,s) \geqslant u^s(t)/u^s(s)$.

Следствие. *Пусть $k > -\dfrac{1}{p'}$, оператор T изотонен, $u_0(t) = t^{-k-1}$ и при $t \in [0,b]$ выполняется неравенство $(Tu_0)(t) \leqslant t^{-k-2}$.*

Тогда $(t/s)^{-k-1} \leqslant C(t,s) \leqslant (t/s)^{-k}$.

Замечание. Функции $(t/s)^{\mu}$, где $\mu \in \{-k-1, -k\}$, являются функциями Коши полуоднородных задач для сингулярных обыкновенных дифференциальных уравнений.

Пусть $\tilde{\mathcal{L}}\colon D^pS(\tau) \to L^p$ — конечномерное линейное расширение [1, стр. 125] оператора $\mathcal{L}$.

Теорема 2. *Пусть существуют функции $u_i > 0$ $(i = 1, 2)$ такие, что $u_i^{\tau} \in D^pS(\tau)$ для любого $\tau \in (0, b)$ и $(-1)^i\tilde{\mathcal{L}}u_i^{\tau} \geqslant 0$.*

Тогда при $t \geqslant s > 0$ справедлива оценка $u_1^s(t)/u_1^s(s) \leqslant C(t, s) \leqslant u_2^s(t)/u_2^s(s)$.

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Об асимптотических формулах для собственных значений и собственных функций системы Дирака

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Рассмотрим оператор Дирака L_Q, порожденный дифференциальным выражением

$$l_Q(\mathbf{y}) = -B\mathbf{y}' + Q\mathbf{y}$$

в пространстве $H = L_2[0, \pi] \oplus L_2[0, \pi] \ni \mathbf{y}$, где

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad Q(x) = \begin{pmatrix} q_1(x) & q_2(x) \\ q_3(x) & q_4(x) \end{pmatrix}, \qquad \mathbf{y}(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},$$

а функции q_j, $j = 1, 2, 3, 4$, предполагаются суммируемыми и комплекснозначными, и регулярными краевыми условиями.

Унитарным преобразованием можно добиться равенства $q_4 = -q_1$ и $q_2 = q_3$, что и будет предполагаться далее.

Краевые условия имеют вид

$$U(\mathbf{y}) = \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ u_{21} & u_{22} & u_{23} & u_{24} \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \\ y_1(\pi) \\ y_2(\pi) \end{pmatrix} = 0,$$

причем условия называются регулярными, если

$$J_{14} + J_{32} \pm i(J_{42} - J_{13}) \neq 0$$

и строго регулярными, если дополнительно $(J_{12}+J_{34})^2 \neq (J_{14}+J_{32})^2+(J_{42}-J_{13})^2$.

Мы расскажем о результатах, связанных с асимптотическим поведением собственных значений и собственных функций оператора L_Q для разных классов потенциалов Q.

Устойчивость линейных гибридных функционально-дифференциальных систем с последействием (ЛГФДСП)[1]

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Исследованию по устойчивости решений ЛГФДСП посвящено сравнительно мало работ. Запишем абстрактную ЛГФДСП в виде

$$\mathcal{L}_{11}x + \mathcal{L}_{12}y = \dot{x} - F_{11}x - F_{12}y = f, \quad \mathcal{L}_{21}x + \mathcal{L}_{22}y = \Delta x - F_{21}x - F_{22}y = g. \quad (1)$$

Здесь и ниже $\mathbb{R}^n$ — пространство векторов $\alpha = \operatorname{col}\{\alpha^1, ..., \alpha^n\}$ с действительными компонентами и с нормой $\|\alpha\|_{\mathbb{R}^n}$. Предполагаем, что функции $f, g, y : [0, \infty) \to \mathbb{R}^n$ принадлежат пространству L локально суммируемых функций с полунормами $\|f\|_{L[0,T]} = \int_0^T \|f(t)\|_{\mathbb{R}^n}\, dt$ для всех $T > 0$, а $x : [0, \infty) \to \mathbb{R}^n$ — пространству D локально абсолютно непрерывных функций с полунормами $\|x\|_{D[0,T]} = \|\dot{x}\|_{L[0,T]} + \|x(0)\|_{\mathbb{R}^n}$ для всех $T > 0$. Операторы $\mathcal{L}_{11}, F_{11} : D \to L$, $\mathcal{L}_{12}, F_{12} : L \to L$, $\mathcal{L}_{21}, F_{21} : D \to L$, $\mathcal{L}_{22}, F_{22} : L \to L$ предполагаются линейными непрерывными и вольтерровыми. Обозначим $(\Delta y)(t) = y(t) - y(t-h)$, если $t \geqslant h > 0$, и $(\Delta y)(t) = y(t)$, если $t \in [0, h)$.

Пусть модельное уравнений $\mathcal{L}_{11}x = z$ и банахово пространство B с элементами из пространства L ($B \subset L$, и это вложение непрерывно) выбраны так, что решения этого уравнения обладают интересующими нас асимптотическими свойствами. Пусть оператор Коши W_{11} для уравнения $\mathcal{L}_{11}x = z$ непрерывно действует из пространства B в пространство B и вольтерров, и пусть столбцы матрицы фундаментальных решений X и $\dot{X}$ принадлежат пространству B. Можно для банахова пространства $B \subset L$ ввести банахово пространство $D(\mathcal{L}_{11}, B)$ с нормой $\|x\|_{D(\mathcal{L}_{11},B)} = \|\mathcal{L}_{11}x\|_B + \|x(0)\|_{\mathbb{R}^n}$. Это пространство линейно изоморфно пространству С.Л.Соболева $W_B^{(1)}([0,\infty))$ с нормой $\|x\|_{W_B^{(1)}([0,\infty))} = \|\dot{x}\|_B + \|x\|_B$. Дальше будем это пространство обозначать W_B. При этом $W_B \subset D$, и вложение непрерывно.

Операторы $\mathcal{L}_{11}$, $\mathcal{L}_{21}$, F_{11}, $F_{21} : D \to L$ рассматриваются как приведения на пару (W_B, B): $\mathcal{L}_{11}$, $\mathcal{L}_{21}$, F_{11}, $F_{21} : W_B \to B$. Операторы Δ, $\mathcal{L}_{12}$, $\mathcal{L}_{22}$, F_{12}, $F_{22} : L \to L$ также рассматриваются как приведения на пару (B, B): Δ, $\mathcal{L}_{12}$, $\mathcal{L}_{22}$, F_{12}, $F_{22} : B \to B$ и предполагаются линейными вольтерровыми и ограниченными.

Поставим задачу, когда для уравнения (1) при любом $\{f, g\} \in B \times B$ для ее решений имеем $\{x, y\} \in W_B \times B$.

Рассмотрим второе уравнение $\mathcal{L}_{21}x + \mathcal{L}_{22}y = g$. Будем считать, что оператор $\mathcal{L}_{22} : B \to B$ вольтеррово обратим, т.е. существует $\mathcal{L}_{22}^{-1} : B \to B$, и оператор $\mathcal{L}_{22}^{-1}$:

[1]Работа поддержана ЗАО «ПРОГНОЗ».

$B \to B$ вольтерров. Тогда это уравнение запишется в виде $\mathcal{L}_{22}^{-1}\mathcal{L}_{21}x + y = \mathcal{L}_{22}^{-1}g$. Выразим y, $y = -\mathcal{L}_{22}^{-1}\mathcal{L}_{21}x + \mathcal{L}_{22}^{-1}g$, и подставим в первое уравнение $\mathcal{L}_{11}x + \mathcal{L}_{12}y = f$: $(\mathcal{L}_{11} - \mathcal{L}_{12}\mathcal{L}_{22}^{-1}\mathcal{L}_{21})x = f - \mathcal{L}_{12}\mathcal{L}_{22}^{-1}g$.

Обозначим $\mathcal{L} = \mathcal{L}_{11} - \mathcal{L}_{12}\mathcal{L}_{22}^{-1}\mathcal{L}_{21}$ и $f_1 = f - \mathcal{L}_{12}\mathcal{L}_{22}^{-1}g$. Получили уравнение $\mathcal{L}x = f_1$. Предположим, что вольтерров оператор $\mathcal{L} : W_B \to B$ вольтеррово обратим, т.е. для уравнения $\mathcal{L}x = f_1$ при любом $f_1 \in B$ выполняется $x \in W_B$ и оператор $\mathcal{L}^{-1} : B \to W_B^0$ вольтерров, где $W_B^0 = \{x \in W_B, x(0) = 0\}$. Таким образом, мы решили задачу, когда для уравнения (1) при любом $\{f, g\} \in B \times B$ его решение $\{x, y\}$ принадлежит $W_B \times B$.

Задача Карлемана для эллиптических систем второго порядка на плоскости[1]

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Хорошо известно, что задача Дирихле корректна для всех эллиптических уравнений на плоскости. Положение меняется при переходе к системам таких уравнений. Еще в 1948 году А. В. Бицадзе привел пример системы вида

$$a_{11}\frac{\partial^2 u}{\partial x^2} + (a_{12} + a_{21})\frac{\partial^2 u}{\partial y^2} + a_{22}\frac{\partial^2 u}{\partial y^2} = 0 \qquad (1)$$

с постоянными и только старшими коэффициентами $a_{ij} \in \mathbf{R}^{l\times l}$, для которой задача Дирихле в единичном круге имеет бесконечное число линейно независимых решений. Позднее им был выделен класс систем, названных слабо связанными, для которых задача Дирихле в области D, ограниченной гладким контуром Γ, фредгольмова. Этот класс можно описать следующим образом. Для каждой эллиптической системы (1) найдутся такие матрицы $b, J \in \mathbb{C}^{l\times l}$, что спектр J лежит в верхней полуплоскости, выполнено матричное равенство $a_0 b + a_1 bJ + a_2 bJ^2 = 0$ и блочная матрица B с элементами $B_{11} = \overline{B_{12}} = b$, $B_{21} = \overline{B_{22}} = bJ$ обратима. При этом любая другая пара (b_1, J_1) с теми же свойствами связана с (b, J) соотношениями $b_1 = bd$, $J_1 = d^{-1}Jd$ с некоторой обратимой матрицей d. В этих обозначениях указанный класс определяется условием $\det b \neq 0$. Можно показать, что этот класс совпадает с системами, удовлетворяющими условию дополнительности по Лопатинскому.

Не носит универсального характера и задача Неймана, которая определяется краевым условием Дирихле по отношению к функциям v, сопряженным к решениям (1). Более точно, эти функции удовлетворят соотношениям

$$\frac{\partial v}{\partial x} = -\left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right), \quad \frac{\partial v}{\partial y} = a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}.$$

Необходимое и достаточное условие фредгольмововсти этой задачи описывается условием $\det(a_{21}b + a_{22}bJ) \neq 0$.

[1]Работа выполнена при поддержке Международного проекта (0113РК01031) Министерства образования и науки Республики Казахстан.

Аналогичная ситуация имеет место и для общей локальной задачи Пуанкаре, которая определяется краевым условием

$$\left(p\frac{\partial u}{\partial x} + q\frac{\partial u}{\partial y}\right)\Big|_\Gamma = f \tag{2}$$

с некоторыми матрицами-функциями $p, q \in C(\Gamma)$. Более точно, для каждой эллиптической системы (1) найдутся такие постоянные матрицы p, q, что задача (2) не является фредгольмовой.

Однако существуют нелокальные краевые задачи, корректные для любых эллиптических систем. Простейшей из них является задача типа Карлемана

$$(u + u \circ \alpha)\big|_\Gamma = f_1, \quad (v - v \circ \alpha)\big|_\Gamma = f_2,$$

где α есть диффеоморфизм контура Γ на себя, меняющий его направление обхода. В работе показано, что эта задача однозначно разрешима для любой эллиптической системы (1).

Задача Коши для уравнения Кана—Хилларда с вязкостью в пространстве равномерно непрерывных ограниченных функций

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Для дифференциального уравнения в частных производных, моделирующего динамику кристаллизации в многокомпонентных сплавах с учетом вязкости:

$$(1 - \alpha)\, u_t - \alpha \Delta u_t = -\Delta^2 u - \Delta u + \Delta u^3,$$

где $\alpha \in]0, 1[$ — известный параметр, $\Delta = \partial^2/x_1^2 + \dots + \partial^2/x_n^2$ — дифференциальный оператор Лапласа, найден временной промежуток существования классического решения задачи Коши и оценка нормы этого решения в пространстве равномерно непрерывных и ограниченных в $\mathbb{R}^n$ функций.

Численно-аналитический метод исследования некоторых линейных функционально-дифференциальных уравнений

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Доклад посвящен результатам исследования скалярного линейного функционально-дифференциального уравнения (ЛФДУ) запаздывающего типа с полиномиальными коэффициентами:

$$\dot{x}(t) = a(t)x(t-1) + b(t)x(t/q) + f(t), \; q > 1.$$

Основное внимание уделяется начальной задаче с начальной точкой, когда начальное условие задается в начальной точке, и ищется классическое решение, подстановка которого в исходное уравнение обращает его в тождество. В качестве метода исследования применяется метод полиномиальных квазирешений [1,2], который основан на представлении неизвестной функции $x(t)$ в виде полинома степени N. При подстановке этой функции в исходное уравнение возникает невязка $\Delta(t) = O(t^N)$, для которой получено точное аналитическое представление. Тогда под полиномиальным квазирешением понимается точное решение в виде полинома степени N возмущенной на невязку исходной начальной задачи. Доказаны теоремы существования у рассматриваемого ЛФДУ полиномиальных квазирешений и точных полиномиальных решений. Приведены результаты численного эксперимента.

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Число гауссовых пакетов на некоторых модельных декорированных графах

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Определим пространство (частный случай так называемого *декорированного графа*), приклеив отрезок к двумерному цилиндру или тору. Рассмотрим задачу Коши для уравнения Шрёдингера на этом пространстве с локализованным начальным условием (*гауссовым пакетом*) (см. [1]). Нас интересует величина $N(T)$ — количество пакетов на отрезке в момент времени T. Эта задача связана с теоретико-числовой задачей об асимптотике числа неравенств вида: линейные комбинации некоторого набора чисел не превосходят T. Она была решена при нахождении энтропии газа Бозе—Маслова (см. статью [2] и ссылки в ней). Используя результаты из [2], мы получили асимптотические формулы для $\log N(T)$.

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Нетеровы краевые задачи для матричных уравнений

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Найдены необходимые и достаточные условия существования решения [1]

$$Z(t) = (z^{(\alpha,\beta)}(t)), \quad Z^{(\alpha,\beta)}(\cdot) \in C^1[a;b\,], \quad \alpha = 1,\ 2,\ \dots\ ,\ m, \quad \beta = 1,\ 2,\ \dots\ ,\ n$$

нетеровой ($m \neq n$) матричной краевой задачи

$$Z'(t) = AZ(t) + Z(t)B + F(t), \quad \mathcal{L}Z(\cdot) = \mathcal{A}, \quad \mathcal{A} \in \mathbb{R}^{m\times n}, \quad F(t) \in \mathbb{C}[a,b]. \qquad (1)$$

Здесь $A \in \mathbb{R}^{m\times m}$ и $B \in \mathbb{R}^{n\times n}$ — постоянные матрицы; $\mathcal{L}Z(\cdot)$ — линейный ограниченный векторный функционал: $\mathcal{L}Z(\cdot)\ :\ C^1[a;b] \to \mathbb{R}^{\ell\times n}$. Как известно [1], общее решение $X(t,\Theta) = U(t)\cdot\Theta\cdot V(t)$, $\Theta \in \mathbb{R}^{n\times n}$, однородной части матричного уравнения (1) определяют нормальные фундаментальные матрицы $U(t)$ и $V(t)$:

$$U'(t) = AU(t), \quad U(a) = I_n, \quad V'(t) = BV(t), \quad V(a) = I_n, \quad \Theta \in \mathbb{R}^{n\times n}.$$

Общее решение $Z(t) \in \mathbb{C}^1[a,b]$ задачи Коши для уравнения (1) имеет вид [2]

$$Z(t,\Theta) = X(t,\Theta) + K[F(s)](t), \quad K[\Phi(s)](t) := \int_a^t U(t)U^{-1}(s)F(s)V(t)V^{-1}(s)\ ds.$$

Определим оператор $\mathcal{M}[\mathcal{B}]:\ \mathbb{R}^{\ell\times n} \to \mathbb{R}^{\ell\cdot n}$, который ставит в соответствие матрице $\mathcal{B} \in \mathbb{R}^{\ell\times n}$ вектор-столбец $\mathcal{M}[\mathcal{B}] \in \mathbb{R}^{\ell\cdot n}$, составленный из n столбцов матрицы $\mathcal{B}$, а также обратный оператор $\mathcal{M}^{-1}\{\mathcal{M}[\mathcal{B}]\}:\ \mathbb{R}^{\ell\cdot n} \to \mathbb{R}^{\ell\times n}$. Пусть $\Xi^{(j)} \in \mathbb{R}^{m\times n}$ — базис пространства $\mathbb{R}^{m\times n}$.

Теорема. *При условии* $P_{\mathcal{Q}^*}M\{\mathcal{A} - \mathcal{L}K[F(s)](\cdot)\} = 0$ *и только при нем общее решение матричной краевой задачи* (1)

$$Z(t,\Theta_r) = X(t,\Theta_r) + G[F(s);\mathcal{A}](t), \quad \Theta_r := \mathcal{M}^{-1}[P_{\mathcal{Q}_r}c_r], \quad c_r \in \mathbb{R}^r$$

определяет обобщенный оператор Грина

$$G[F(s);\mathcal{A}](t) := X\{t, \mathcal{M}^{-1}\{\mathcal{Q}^+M[\mathcal{A} - \mathcal{L}K[F(s)](\cdot)]\}\} + K[F(s)](t).$$

Здесь $P_{\mathcal{Q}^*}$ *— ортопроектор* $\mathbb{R}^{\ell\cdot n} \to \mathbb{N}(\mathcal{Q}^*)$; *матрица* $P_{\mathcal{Q}_r}$ *составлена из* r *линейно независимых столбцов ортопроектора* $P_{\mathcal{Q}}:\mathbb{R}^{m\cdot n\times m\cdot n} \to \mathbb{N}(\mathcal{Q})$, *где*

$$\mathcal{Q} := [\ M[\mathcal{Q}^{(1)}]\ \dots\ M[\mathcal{Q}^{(m\cdot n)}]\ ] \in \mathbb{R}^{\ell\cdot n\times m\cdot n}, \quad \mathcal{Q}^{(j)} := \mathcal{L}_i U(\cdot)\Xi^{(j)}V(\cdot) \in \mathbb{R}^{\ell\times n}.$$

Утверждение доказанной теоремы является обобщением соответствующих утверждений [2] на случай нетеровой краевой задачи (1).

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Contents

Addendum

Fractional Minimal Surfaces

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Fractional minimal surfaces, introduced by Caffarelli and Souganidis and further studied by Caffarelli, Roquejoffre and Savin, appear naturally in many contexts such as threshold dynamics, when local interactions are replaced by long range interactions. Unlike classical minimal surfaces, there are no known representation formulas and ODE techniques are not applicable. Hence, although they can be constructed variationally, it was not known whether a catenoidal fractional minimal surfaces exists. We present a construction of fractional minimal surfaces similar to catenoid, or parallel planes.

This is joint work with Manuel del Pino and Juncheng Wei.

Differential Equations with Nonlinear by the Unknown Degenerated Operator before the Derivative

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In the article [1] the theory of Jordan chains for multiparameter operator-functions $A(\lambda) : E_1 \to E_2$, $\lambda \in \Lambda$, $\dim\Lambda = k$, $\dim E_1 = \dim E_2 = n$ is developed, where $A_0 = A(0)$ is a degenerated operator, $dim Ker A_0 = 1$, $Ker A_0 = \{\varphi\}$, $Ker \mathrm{A}_0^* = \{\psi\}$ and the operator-function $A(\lambda)$ is supposed to be linear on λ. Applications to degenerate differential equations of the form $[A_0 + R(\cdot, x)]x' = Bx$ were given.

In this article the relevant results for the differential equations with nonlinearly depended on unknown variable x are considered. The theory of singular points for such degenerated systems of the differential equations is developed and bifurcation theorems in their neighborhoods are proved.

Note the possible applications of differential equations can be found in transonic aerodynamics [2].

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Nondegeneracy of Nonradial Nodal Solutions to Yamabe Problem

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We provide the first example of a sequence of *nondegenerate* in the sense of Duyckaerts–Kenig–Merle, nodal nonradial solutions to the critical Yamabe problem

$$-\Delta Q = |Q|^{\frac{2}{n-2}} Q, \quad Q \in \mathcal{D}^{1,2}(\mathbb{R}^n).$$

This is a joint work with J. Wei.

Feynman's Formulas and Averaging Semigroups

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To obtain representations of semigroups generated by Schrödinger operators in Hilbert space, some new averaging procedure sets of semigroups V_s, $s \in S$, by measure on the set S have been used. Although the average value F of the sets of semigroups is not a semigroup, but it is equivalent by Chernoff [1] to a semigroup U, determined by the mean value F by using the Feynman formula. Feynman's formula is called [2] the representation of the semigroup or group of Schrödinger in the space $L_2(Q)$ using the limit of integrals over Cartesian powers space Q coordinate as multiplicity tends to infinity, which takes the form $U(t) = \lim_{n\to\infty} [F(t/n)]^n$, $t \geqslant 0$. Generator of semigroup U obtained by the Feynman formula, which is naturally defined as the mean value averaged generators of semigroups, is one of Schrödinger operators. Defined in this way, the averaging procedure for self-adjoint operators is an extension of usual averaging procedure in the Banach space of bounded linear operators, generalizing it to the case of unbounded operators.

This procedure will be applied to obtain representations of groups and semigroups generated by a one-dimensional Schrödinger operator on the branched manifold (graph) using the Feynman formulas. Operation of differentiation functions uniquely defined for functions defined on a domain or on a smooth manifold, but it needs to be defined for functions defined on manifolds containing the branch point. For this purpose, given description of the set self-adjoint extensions of a symmetric operator, originally defined on smooth function whose support does not contain branch points of the graph [3]. Representations of semigroups generated by Schrödinger operators on the graph obtained by Feynman formulas and procedures of averaging generators of semigroups.

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On the Threshold Behaviour of Waveguide Scattering Matrices

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A waveguide occupies a domain G in $\mathbb{R}^d$, $d \geqslant 2$, having several cylindrical outlets to infinity. The waveguide is described by the

Dirichlet problem for the Helmholtz equation. The scattering matrix $S(\mu)$ with spectral parameter μ changes its size when μ crosses a threshold. To study the behaviour of $S(\mu)$ in a neighborhood of a threshold, we introduce an "augmented" scattering matrix $\mathcal{S}(\mu)$ that keeps its size near the threshold, where the matrix $\mathcal{S}(\mu)$ is analytic in μ. Deriving a connection between the matrices $S(\mu)$ and $\mathcal{S}(\mu)$ we prove the existence of the right and the left one-sided limits of the scattering matrix $S(\mu)$ and calculate the limits in terms of the augmented matrix $\mathcal{S}(\mu)$.

The report is based on the joint work with B. A. Plamenevskii and O. V. Sarafanov [1].

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Pseudo-Parabolic Regularization of Forward-Backward Parabolic Equations

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We study the initial-boundary value problem

$$\begin{cases} u_t = [\varphi(u)]_{xx} + \varepsilon[\psi(u)]_{txx} & \text{in } Q := \Omega \times (0,T] \\ \varphi(u) + \varepsilon[\psi(u)]_t = 0 & \text{in } \partial\Omega \times (0,T] \\ u = u_0 \geqslant 0 & \text{in } \Omega \times \{0\}\,, \end{cases}$$

with *Radon measure-valued initial data*, under various assumptions on the regularizing term ψ which is increasing and bounded. The results we will discuss stem from recent joint papers with M. Bertsch and F. Smarrazzo.

О многомерных решениях уравнения $(xy)x = x(yx)$ со свойством универсальности

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Исследуется изотопически инвариантное многообразие E гладких луп, определяемых тождеством эластичности $(xy)x = x(yx)$ (лупы E). Одним из авторов было ранее доказано, что это многообразие

а) является подмногообразием средних луп Бола;

б) включает в себя многообразие луп Муфанг;

в) существует всего два класса трехмерных луп E, E_1 и E_2 (двумерные лупы E являются группами и не представляют интереса).

Помимо указанных классов к настоящему времени известны только еще 3 класса четырехмерных луп E.

В нашей работе классы E_1 и E_2 обобщаются на многомерный случай. В исследовании существенную роль играет тот факт, что каждому классу луп однозначно соответствует класс многомерных три-тканей. Это дает возможность записать в инвариантной форме дифференциальные уравнения проблемы. В терминах теории тканей исследуемый класс луп характеризуется тем, что у соответствующей ткани тензор кручения имеет ранг 1 (то есть определяемая им алгебра имеет одномерную производную алгебру). Полученную систему уравнений удалось проинтегрировать в самом общем случае и найти уравнения искомых луп в некоторых локальных координатах.

Об аналитических решениях уравнения $(x/y)(z\backslash x) = x((zy)\backslash x)$

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Известны различные обобщения групп Ли, имеющие разнообразные приложения, — это аналитические лупы Муфанг, левые, правые и средние лупы Бола, в которых (вместо ассоциативности) выполняются соответственно универсальные тождества $(xy)(zx) = x((yz)x)$, $(x(yx))z) = x(y(xz))$, $((xy)z)y = x((yz)y)$, $(x/y)(z\backslash x) = x((zy)\backslash x)$. Каждому многообразию луп однозначно соответствует класс многомерных три-тканей.

В докладе рассматривается многообразие средних луп Бола, для которых тензор кривизны соответствующей средней ткани Бола является ковариантно постоянным относительно канонической связности Черна. Используя аппарат теории тканей, мы записываем дифференциальные уравнения проблемы в инвариантной форме и анализируем их. При некоторых естественных дополнительных условиях эти уравнения удается проинтегрировать и найти уравнения средних луп Бола в локальных координатах.